R. A. Lindsay

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ELECTRIC OSCILLATIONS
AND
ELECTRIC WAVES

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WITH APPLICATION TO RADIOTELEGRAPHY AND INCIDENTAL APPLICATION TO TELEPHONY AND OPTICS

BY

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PREFACE

This book is designed to present a mathematical treatment of some of the fundamentals of the theory of electric oscillations and electric waves.

Although the selection of material particularly applicable to radiotelegraphy has been the first consideration, yet, because the electromagnetic theory, which is fundamental to radiotelegraphy, is fundamental also to optics, wire telephony and power transmission, it is hoped that the volume may be useful in these fields also.

Book I on Electric Oscillations and Book II on Electric Waves are practically independent, so that a reader with a fair knowledge of mathematics may read the two books in either sequence.

A student in optics might confine his attention almost entirely to Book II. A mature reader primarily interested in wire telephony or power transmission might begin at Chapter XVI of Book I, and continue through Chapter XVII, with such occasional references to the earlier chapters as are necessary for familiarity with the methods employed. He might then look into some of the earlier chapters in order to acquaint himself with the various transformer problems arising in connection with coupled circuits.

It is suggested that students of radiotelegraphy begin at the beginning of Book I and read the various chapters consecutively, with the possible exception of Chapters IX, X, and XV, which may be omitted or postponed without rendering difficult the understanding of what follows. It is perhaps unnecessary to say that the theoretical work of this book should be supplemented by copious descriptive matter and laboratory work.

Certain important subjects related to radiotelegraphy have been omitted—particularly the matter of spark-gaps, arcs, vacuum tubes, direction finders and the propagation of electric waves over the surface of the earth. These defects are to be partly remedied by including the omitted material in a revision

of the author's earlier book "The Principles of Wireless Telegraphy" and in other writings now in preparation.

The writer takes pleasure in acknowledging his indebtedness to Mr. Yu Ching Wen for valuable assistance in reading the proofs; to Mr. H E. Rawson for supplying a draftsman; and to the publishers for their accuracy and dispatch in the difficult composition and manufacture of the book.

G. W. P.

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Book I

ELECTRIC OSCILLATIONS

CHAPTER I

FUNDAMENTAL LAWS AND EQUATIONS

1. Notation.—

I = Current (constant),
Q = Quantity of electricity (constant),
E = E.m.f., or difference of potential (constant),
i = Instantaneous value of current at time t (variable),
q = Instantaneous value of quantity at time t (variable),
e = Instantaneous value of e.m.f. at time t (variable),
R = Resistance,
L = Self-inductance,
C = Capacity.

When several of these quantities enter into the same equation, they must all be measured in some common set of units.

2. Kirchhoff's Current Law[1] for a Steady State.—When a conductor is traversed by a constant current and all the charges

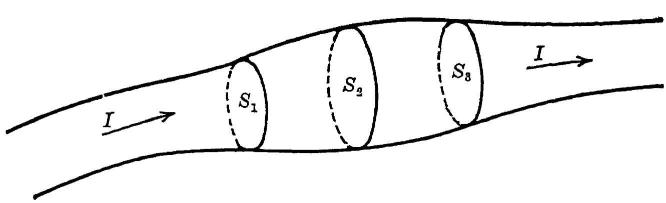


Fig. 1.—Conductor with sections S_1, S_2, S_3

of the conductor are constant, the same amount of electricity per second (*i.e.*, the same current) flows through every cross section, S_1, S_2, S_3 of the conductor (Fig. 1). In this figure the two lines in a general horizontal direction are boundaries of

[1] Kirchhoff, *Pogg. Ann.*, Vol. 72 (1847). Also *Gesammelte Abhandlungen.*

the conductor across which no current is allowed to flow. Any two transverse surfaces, as S_1 and S_3, together with the boundaries of the conductor, enclose a region of the conductor. Now if more or less electricity flows per second into the region through S_1 than flows out through S_3 in the same time, there will be a growth or a decrease of electric charge within the region, which is contrary to the hypothesis of a steady state. Therefore, the current in through any surface S_1 and out through any surface S_3 must be the same if the state of current and charge is constant in time.

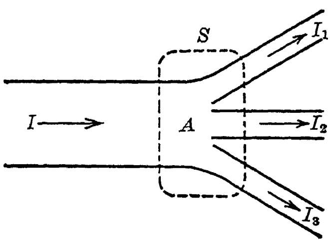


FIG. 2—Branched conductor, with region about A enclosed by a surface S.

If an electric conductor is branched as at A in Fig. 2, so that through the main conductor a current I flows into any surface S enclosing A, while currents I_1, I_2, I_3, . . . flow out of S through the branches, and if there is no growing accumulation of electricity within the enclosure, then numerically,

$$(I)_{in} = (I_1 + I_2 + I_3 + \ . \ . \ . \ )_{out} \tag{1}$$

If now we give algebraic sign to currents, attributing to currents "out" a sign opposite to currents "in," then

$$I + I_1 + I_2 + I_3 + \ . \ . \qquad = 0; \tag{2}$$

that is,

$$\Sigma_S I = 0, \tag{3}$$

where Σ_S indicates algebraic summation applied to all parts of the closed surface S.

Equations (1), (2), and (3) are merely different methods of stating symbolically that electricity is *conserved*, and that in the cases under consideration there is no accumulating of electricity within a certain region, and that, therefore, the amount of electricity flowing out of the region in a given time is equal to the amount flowing into the region in the same time.

3. Kirchhoff's Current Law in the above Form Inapplicable at Regions Containing Capacity.—Fig. 3 represents an electric circuit containing a condenser C. If we suppose that a battery or other source of e.m.f. is applied at B, current will flow for a short time into the condenser. If now we draw a closed surface

S including one plate of the condenser, it is apparent that there may be an electric current i flowing into the region bounded by the surface S, while there is at the same time no current (in the elementary sense of the word) flowing out of the region bounded by S.

As another example, if we suppose a conducting wire AB, Fig. 4, to be supported on insulators and open-ended at B, and let a battery be connected between the other end A and the ground E, it is apparent, according to elementary notions, that a charge of electricity will flow into the conductor at A,—this charge constituting an electric current i_1 at A—while there

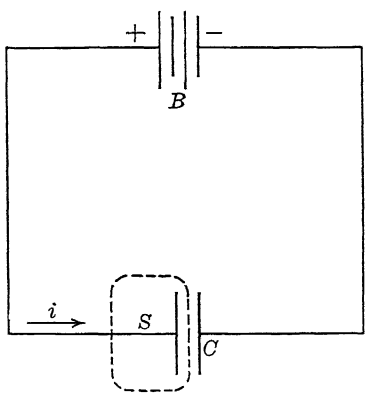


FIG. 3.—Circuit containing battery B and condenser C, with one plate enclosed by surface S.

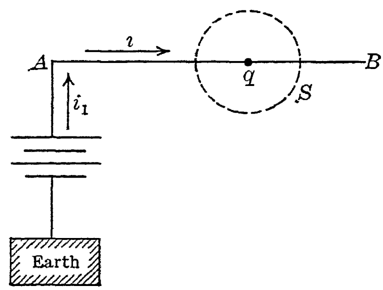


FIG. 4.—Illustrating distributed capacity

is no current out from the end B of the conductor. At any point intermediate between A and B, there will in general be a current i (say), and this current will be different at different points along the conductor; so that if a closed surface S be drawn, there will in general be a difference between the current flowing *into* and the current flowing *out of* S.

4. Generalization of Kirchhoff's Current Law so as to Apply to Variable Currents. Law of Conservation of Electricity.—If i_i is the instantaneous value of current flowing into a given region bounded by a closed surface S, and i_o is the instantaneous value of the current at the same instant flowing out of the region S, we may suppose that in the time dt the current into the region delivers a charge $i_i dt$ and the current out from the region carries away a charge $i_0 dt$; the difference between these two quantities

is dq (say), where dq is the gain of charge of the region in the time dt. The assumption that there is no creation or destruction of electricity during the process makes

$$dq = i_i dt - i_0 dt; \tag{4}$$

therefore,

$$\frac{dq}{dt} = i_i - i_0, \tag{5}$$

As a generalization of this equation, if we consider current flowing into a given region bounded by a closed surface to be positive, and current flowing out to be negative, then

$$\frac{dq}{dt} = i_i. \tag{6}$$

Equation (6) may be stated as follows: *The excess of the current flowing into a given region at a given time over the current flowing out at the same time is the time-rate of increase of quantity of electricity within the region at that time.*

This is a statement of the Law of the Conservation of Electricity, and applies to all cases of the flow of electricity whether the flow is constant or variable. We shall call the equation *Kirchhoff's Generalized Current Law*, or *Kirchhoff's Current Law.* The terms employed in the statement and equations are explained in the next section.

5. Explanation of Terms of Foregoing Statements and Equations. Intrinsic Charge.—The quantities q and i_i in equation (6) must be measured in the same set of units. If i_i is in amperes, q must be in coulombs. If, on the other hand, i_i is in absolute units of either the electrostatic or the electromagnetic system, q must be in absolute units of the same system.[1]

The charge indicated by q is a charge that can be increased or diminished only by an actual transfer of electricity (free electrons) into the region containing q. Such a charge is known as an *intrinsic* charge, and is to be distinguished from certain *induced* charges to be considered later.

The current i_i must include any actual transfer of electricity into the region, whether of the ordinary *conduction* variety or whether the transfer is by an actual transfer of charged matter into the region; that is, i_i must include *conduction* and *convection* currents of electricity. It is highly probable that all transfer of

[1] For discussion of these units see PIERCE, "Principles of Wireless Telegraphy," Appendix I.

free electricity, even in metallic conduction, is accompanied by the flow of matter in the form of electrons, and is, therefore, essentially a convection current; but this subject may properly be deferred to later consideration. The current i_i, however, in the present form of the equation does not include *displacement currents* to be treated in Book II.

6. Generalization of Kirchhoff's Electromotive Force Law.— If we have a circuit of the form of Fig. 5 in which an e.m.f. e is applied to a constant resistance R, a constant inductance L, and a constant capacity C in series, the instantaneous value of

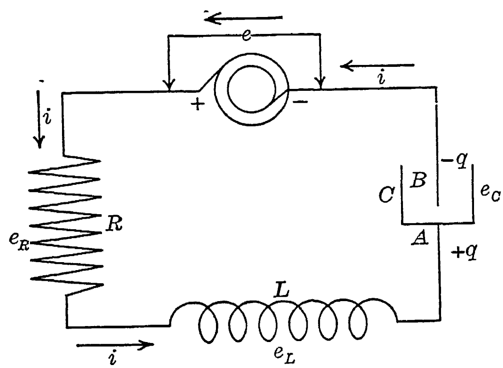


FIG. 5—Circuit containing e. m. f. e, resistance R, self-inductance L, and capacity C.

the e.m.f. e at any time t is equal to the sum of the instantaneous values of the potential drops e_R, e_L, e_C; that is,

$$e = e_R + e_L + e_C, \tag{7}$$

in which

e_R = the fall of potential in the resistance R,
e_L = the fall of potential in the inductance L,
e_C = the fall of potential in the capacity C.

Let us now adopt the following *rule of signs:* If e and i are in the direction of the arrows they are given a *positive* sign. If they are in the opposite direction they are given a *negative* sign. If the charge on the plate A (toward which positive i flows) is *positive* q is *positive*. If this charge is negative q is negative.

Then by Ohm's law,[1] the fall of potential in the resistance R is

$$e_R = Ri, \tag{8}$$

[1] G. S. Ohm, "Die galvanische Kette mathematisch bearbeitet," Berlin, 1827.

where i is the instantaneous current through the resistance R.

The fall of potential in the inductance L is

$$e_L = L\frac{di}{dt}, \tag{9}$$

where L is the self-inductance of the coil L, and i is the current through L. This current is the same as the current through R, since there is assumed to be no capacity and therefore no accumulation of charge within R and L.

The fall of potential in the condenser C is

$$e_C = \frac{q}{C}, \tag{10}$$

where $+q$ is the charge on the plate A of the condenser, and C is the capacity of the condenser. It is to be noted that there is an equivalent charge of the opposite sign $(-q)$ on the plate B; because, since there is no other capacity in the circuit, the current throughout the circuit at the time t is everywhere the same except within the dielectric of the condenser: and, therefore, the current out of the condenser at B is always the same as the current into the condenser at A, and hence the deficit of charge (the negative charge) of B is always the same as the excess of charge (the positive charge) of A.

By Kirchhoff's Current Law (eq. 6)

$$\frac{dq}{dt} = i;$$

therefore,

$$q = \int i\,dt. \tag{11}$$

If now we substitute (8), (9), (10) and (11) in (7), **we obtain**

$$e = Ri + L\frac{di}{dt} + \frac{\int i\,dt}{C}. \tag{12}$$

In this equation the applied e.m.f. e is usually called the impressed e.m.f. This impressed e.m.f. may be variable, constant or zero. If it is variable its instantaneous value at any time t is to be taken, and the current i is the instantaneous value of the current at the same time t.

It may not be apparent just why the e.m.f. e, represented in Fig. 5 as produced by a dynamo, shall be considered as impressed e.m.f., while the other terms of the equation (12) are regarded as falls of potential. The reply is, that e is the *terminal voltage*

of the dynamo and is, therefore, impressed by a source of power external to the sequence of elements R, L and C. If e is not the terminal voltage of the dynamo, but the total e.m.f. generated in the dynamo, then equation (12) would still be true if we add the resistance of the dynamo to R and add the inductance of the dynamo to L, although in this case difficulty would arise because the equation presupposes a constant L, which would not be the case if the dynamo contained iron in its armature.

As a further note on impressed e.m.f., if we regard e as the terminal voltage of the dynamo, it is evident that we may regard the quantity $e - L\dfrac{di}{dt} - \dfrac{\int idt}{C}$ as the e.m.f. impressed on R; for there is a terminal dynamo voltage e impressed on the circuit; this is opposed by the counter e.m.f. $L\dfrac{di}{dt}$ due to the magnetic field of the self inductance coil L and by the counter e.m.f. $\dfrac{\int idt}{C}$ due to the charge of the condenser, leaving $e - L\dfrac{di}{dt} - \dfrac{\int idt}{C}$ as the e.m.f. impressed on R.

It is perhaps still more instructive to transpose also the term Ri to the left hand side of equation (12), giving

$$e - Ri - L\frac{di}{dt} - \frac{\int idt}{C} = 0 \tag{13}$$

We may now regard Ri as the counter e.m.f. of the resistance, and may interpret equation (13) as an algebraic statement of the fact *that the impressed e.m.f. and the counter e.m.f.'s constitute a system in equilibrium.*

If we have several dynamos or batteries of terminal voltages $e_1, e_2, e_3 \ldots$, these e.m.f.'s being estimated positive when tending to send currents in the direction of the arrows and negative when tending to send currents in the opposite direction, and if we have several capacityless resistances[1] $R_1, R_2, R_3 \ldots$, several capacityless inductances $L_1, L_2, L_3 \ldots$, and several condensers of capacities $C_1, C_2, C_3 \ldots$, all in series, we shall have

$$e_1 + e_2 + e_3 + \ldots - (R_1 + R_2 + R_3 + \ldots)i -$$
$$(L_1 + L_2 + L_3 + \ldots)\frac{di}{dt} - \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots\right)\int idt = 0$$

[1] Some or all of the resistances may be in whole or part the resistances of the inductance coils.

or

$$\Sigma e - (\Sigma R)i - (\Sigma L)\frac{di}{dt} - \left(\Sigma\frac{1}{C}\right)\int i\,dt = 0. \qquad (14)$$

Equation (14) presupposes that the L's, R's, and C's are independent of the time t. It may readily be seen how the equation is to be modified to make it applicable to cases in which these coefficients are variables. We shall, however, have occasion to discuss chiefly those problems in which R, L, and C are constants independent of current i and independent of time t, and shall at present limit ourselves to these conditions. The group of results constituting Kirchhoff's Generalized Electromotive Force Law, or Kirchhoff's Second Law, may be summarized as follows:

7. Summary of Kirchhoff's E.M.F. Law:

1. When there is an instantaneous current i flowing in a constant capacityless and inductanceless resistance R at the time t, there is impressed at the same time at the terminals of the resistance by some source of power external to the resistance a difference of potential e_R equal to Ri and in the direction of i;

2. When there is at the time t an instantaneous current i flowing in a constant capacityless inductance L of resistance R_L, there is impressed at the same time at the terminals of the inductance by some source of power external to the inductance a difference of potential e_L equal to $R_Li + L\frac{di}{dt}$ and in the direction of i;

3. When there is at the time t an instantaneous current i flowing into the positively charged plate of a condenser of constant capacity C, there is an equal current i flowing away from the other plate[1] of the condenser, and there is impressed upon the condenser from some source of power external to the condenser a difference of potential between the plates of the value e_C equal to $\frac{\int i\,dt}{C}$ and in the direction of i;

4. When several of these elements (resistance, inductance and condensers) are in series the total instantaneous impressed e.m.f. is equal to the sum of the instantaneous e.m.f.'s impressed on the elements.

[1] Care must be exercised in determining what is the other plate of the condenser. It is the aggregate of all bodies on which terminate lines of static force from the first plate.

CHAPTER II

THE FLOW OF ELECTRICITY IN A CIRCUIT CONTAINING RESISTANCE, SELF-INDUCTANCE, AND CAPACITY. DISCHARGE, CHARGE, AND CURRENT INTERRUPTION

8. Notation.—

R = Resistance,
L = Self inductance,
C = Capacity,
I = Initial constant current,
E = Constant impressed e.m.f.,
E_0 = Initial difference of potential between the plates of a condenser,
Q_0 = Initial charge on one plate of a condenser prechosen as positive,
Q = Final charges on this plate,
q = Charge at the time t on the condenser plates, A (Fig. 1), prechosen as positive,
i = Instantaneous current flowing toward the plate A at the time t,
e = Impressed e.m.f. at the time t. Let the positive direction of e be toward that plate of the condenser designated as positive.

As we proceed we shall need also the following additional abbreviations:

(i) $$k_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

(ii) $$k_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

(iii) $$\omega_h = +\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

(iv) $$\omega = +\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}},$$

(v) $$a = R/2L.$$

Among these expressions the following algebraic relations are seen to exist

(vi) $$k_1 = -a + \omega_h = -a + j\omega,$$

(vii) $$k_2 = -a - \omega_h = -a - j\omega,$$

(viii) $$k_1k_2 = a^2 - \omega_h{}^2 = a^2 + \omega^2 = \frac{1}{LC},$$

(ix) $$k_1 - k_2 = 2\omega_h = 2j\omega,$$

(x) $$j = \sqrt{-1}.$$

As these relations occur in the text, we shall refer to them by their respective Roman Numerals.

9. Differential Equation of Current and Quantity.—If in a circuit of the form of Fig. 1, we equate the impressed e.m.f. e to the sum of the counter e.m.f.'s (that is, the counter e.m.f. of

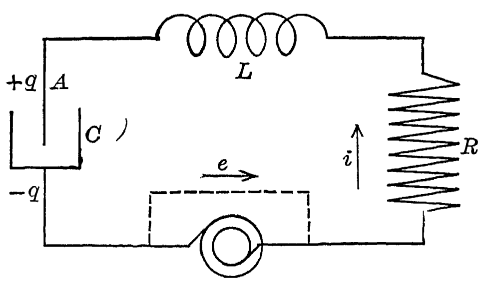


FIG. 1 —Circuit containing R, L, C and impressed e.m.f. e

self inductance + the counter e.m.f. of resistance + the counter e.m.f. of capacity) we have, by (7), (8), (9), and (10) of Chapter I,

$$e = L\frac{di}{dt} + Ri + \frac{q}{C}. \tag{1}$$

We have also the following relation of i to q (6), Chapter I,

$$i = \frac{dq}{dt}, \tag{2}$$

or

$$q = \int i dt. \tag{3}$$

Differentiating (1) and introducing (2), we obtain

$$\frac{de}{dt} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}. \tag{4}$$

Likewise, if we replace i in equation (1) by its value in terms of q from equation (2), we obtain

$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \tag{5}$$

Equations (4) and (5) are the differential equations for the current in the circuit and for the charge in the condenser at any time t in terms of the e.m.f. impressed upon the circuit.

10. General Solution.—A general solution of equations of the form of (4) and (5) is given in Appendix I, Note 6. Instead of making direct use of the solution there given, it is instructive to solve (4) and (5) by elementary methods for specific values of e such as arise in important practical cases.

11. Important Special Problems.—By assigning different values to the impressed e.m.f., e, various special problems arise in connection with the flow of current in condenser circuits. The following of these problems are highly important and interesting:

I. To find i and q during the discharging of a condenser initially charged.

II. To find i and q during the charging of a condenser under a constant impressed e.m.f.

III. To find i and q produced by interrupting a current flowing in an inductance which is shunted by a condenser.

IV. To find i and q under the action of a sinusoidal impressed e.m.f.

These problems will be treated in order (the first three in this chapter, and the fourth in Chapter V). Each problem will be examined in detail, partly on account of the interest that it presents in itself, and partly as introductory to other matter.

I. THE DISCHARGING OF A CONDENSER[1]

12. Differential Equation for Current and Quantity During Discharge.—Suppose a condenser of Capacity C, Fig. 2, to be initially charged with a quantity of electricity $+ Q_0$ on one plate and $- Q_0$ on the other plate, and at the time $t = 0$, let the gap G be closed in such a way that there is no spark[2] at G, then we have the initial conditions.

[1] This problem was first solved by Sir Wm. Thomson, *Phil. Mag.*, 5, p. 393, 1853.

[2] Because a spark has a resistance that is a function of the current through the spark.

When $$t = 0,\ q = Q_0 = CE_0, \tag{6}$$

where E is the initial difference of potential between the plates of the condenser. We have also:

When $$t = 0,\ i = 0. \tag{7}$$

In addition to these initial conditions, we have the fact that the e m.f. impressed upon the circuit is zero; whence the differential equations (4) and (5) take the respective forms

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}, \tag{8}$$

$$0 = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \tag{9}$$

It is seen that (8) *and* (9) *are identical in form. They are the differential equations for the current i and the quantity q during the condenser discharge.*

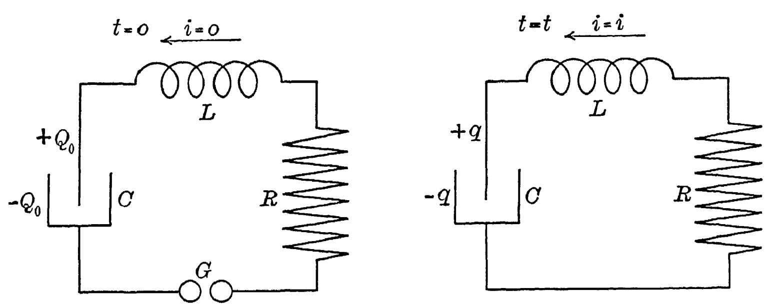


FIG 2 —Illustrating condenser discharge. Left-hand diagram is the condition at $t = o$; right-hand, at $t = t$.

13. General Solution of Equations (8) and (9).—Let us fix our attention upon equation (8). This equation is a *homogeneous linear differential* equation of the *second* order, with constant coefficients. This terminology, which is used generally in the theory of differential equations, has the following significance.

If we regard i and its derivatives $\left(i, \frac{di}{dt}, \frac{d^2i}{dt^2}, \ldots\right)$ as the elements of the equation, the equation is linear in these elements, since products or squares or higher powers of these elements do not enter. It is homogeneous, since every significant term of the equation contains one of the elements to the same power; namely, the first power. It has the constant coefficients L, R, and $1/C$.

The equation is of the second order, by which is meant that the highest order of any derivative is the second order.

The following general propositions in the theory of differential equations are applicable to the problem:

I. The sum of two or more solutions of a linear homogeneous equation is a solution of the equation; that is, the solutions are additive.

II. If we can in any way find a solution of a linear, homogeneous equation of the nth order, the solution, if it contains n independent arbitrary constants, is the most general solution, or the complete integral of the equation.

The proofs of these two propositions are found in Appendix I, Notes 1 and 4. We shall employ the propositions in obtaining the solution of equations (8) and (9).

In the beginning let us attempt to find by inspection a particular solution of (8). We may try anything we like in the search for a solution; for example, let us try $i = A$, a constant. This substituted in (8) yields $0 = 0 + 0 + A/C$, and, therefore, $A = 0$, and $i = 0$. Such a value will not contribute anything by addition to any other solution that may be found.

We might make various other random attempts to find a particular solution of (8), but we shall make greater progress by basing our attempts upon some rational experience, particularly upon experience with the use of exponential functions.

Let us try

$$i = A\epsilon^{kt}, \tag{10}$$

where A and k are constants and ϵ is the base of the natural logarithms. This value of i substituted in (8) gives,

$$0 = \{Lk^2 + Rk + 1/C\}A\epsilon^{kt}. \tag{11}$$

It is seen that we may divide out $A\epsilon^{kt}$ from (11), obtaining

$$0 = Lk^2 + Rk + 1/C. \tag{12}$$

We have thus the result that (10) *is a solution of* (8) *provided k satisfies the quadratic equation* (12).

Solving this quadratic equation (12) for k, we find the two roots

$$k = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = k_1 \qquad \text{by (i), Art. 8,} \tag{13}$$

and

$$k = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = k_2 \qquad \text{by (ii), Art. 8.} \tag{14}$$

Equations (13) and (14) give two specific values of k either of which will make the exponential value of i given in (10) satisfy the differential equation (8).

The coefficient A of equation (10) is entirely arbitrary and may have any values whatsoever so far as may be determined by the given differential equation. The constant k is determined by (13) and (14).

In the attempt to find one particular solution of (8) we have really found two particular solutions, namely, either

$$i = A_1\epsilon^{k_1t}, \tag{15}$$

or

$$i = A_2\epsilon^{k_2t}. \tag{16}$$

In these equations A_1 and A_2 are arbitrary constants, which are in general independent of each other.

Now by Proposition I, Art. **13**, the sum of these two solutions is a solution. That is,

$$i = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t} \tag{17}$$

is a solution of equation (8). In fact, this is the most general solution, or complete integral, of (8), provided k_1 and k_2 are different quantities; for then A_1 and A_2 are two independent arbitrary constants; and by Proposition II, Art. **13**, such a solution is general.

If on the other hand $k_1 = k_2$, the solution (17) reduces to

$$i = (A_1 + A_2)\,\epsilon^{k_1t}, \tag{18}$$

and, therefore, possesses only one arbitrary constant; for the sum of A_1 and A_2 is no more arbitrary than A_1 alone.

The exceptional case with k_1 equal to k_2 arises when

$$R^2 = 4L/C,$$

or

$$\omega_h = \omega = 0,$$

as may be seen by reference to (13) and (14) and to (iii) and (iv), Art. **8**. This is called the *Critical Case*. The critical case requires a special treatment, which is given in Appendix I, Note 7, where the result is obtained in the form of

$$i = (A_1 + A_2t)\,\epsilon^{-\frac{Rt}{2L}} \tag{19}$$

If the reader does not care to follow the reasoning of the Note 7 in Appendix I, he can satisfy himself that (19) is a solution of (8) in the critical case by substituting (19) directly in (8) and introducing also the condition

$$R^2 = 4L/C.$$

Since (19) contains two independent arbitrary constants, it is the complete integral for the critical case.

To sum up, we have found the general solution of (8) to be

$$i = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t}, \qquad \text{provided } R^2 \neq 4L/C, \tag{20}$$

$$i = (A_1 + A_2t)\,\epsilon^{-\frac{Rt}{2L}}, \text{ provided } R^2 = 4L/C. \tag{21}$$

Now to obtain the value of q we may solve directly equation (9), just as we have solved (8). We shall, however, adopt the alternative method of obtaining q by integrating (20) and (21), employing the relation

$$q = \int i\,dt. \tag{22}$$

Equation (20) gives

$$q = \frac{A_1}{k_1}\epsilon^{k_1t} + \frac{A_2}{k_2}\epsilon^{k_2t}, \text{ provided } R^2 \neq 4L/C, \tag{23}$$

and equation (21) gives

$$q = \int (A_1 + A_2t)\,\epsilon^{-at}\,dt,$$

where by (v) $a = R/2L$; whence

$$q = \frac{A_1}{-a}\epsilon^{-at} + A_2\int t\,\epsilon^{-at}\,dt.$$

When the last term of this equation is integrated by parts, we obtain

$$q = \left\{-\left(\frac{A_1}{a} + \frac{A_2}{a^2}\right) - \frac{A_2}{a}t\right\}\epsilon^{-at}, \text{ provided } R^2 = 4L/C. \tag{24}$$

Equations (20), (21), (23), *and* (24) *are the general solutions of the differential equations* (8) *and* (9). *In these equations A_1 and A_2 are arbitrary constants; while k_1, k_2 and a are constants of the circuits defined in equations* (i), (ii), *and* (v), *Art.* 8, *respectively.*

14. Determination of the Arbitrary Constants A_1 and A_2 Subject to the Initial Conditions.—We may now determine the arbitrary constants subject to the initial conditions written above as (7) and (6). These initial conditions are:

When $$t = 0,\ i = 0,$$

and when $$t = 0,\ q = Q_0 = CE_0.$$

In the non-critical case ($R^2 \neq 4L/C$), these initial conditions substituted in (20) and (23) give

$$0 = A_1 + A_2, \tag{25}$$

$$Q_0 = \frac{A_1}{k_1} + \frac{A_2}{k_2}, \tag{26}$$

whence

$$Q_0 = A_1 \left(\frac{1}{k_1} - \frac{1}{k_2}\right)$$

$$= A_1 \left(\frac{k_2 - k_1}{k_1 k_2}\right)$$

This last equation, by (viii) and (ix), Art. **8**, gives

$$Q_0 = A_1 (-2\omega_h LC).$$

Therefore

$$A_2 = -A_1 = \frac{Q_0}{2\omega_h LC} = \frac{E_0}{2\omega_h L}, \text{ provided } R^2 \neq 4L/C, \tag{27}$$

where

$E_0 = Q_0/C =$ initial difference of potential of the plates of the condenser.

In the critical case ($R^2 = 4L/C$), the substitution of the initial conditions into (21) and (24) gives

$$0 = A_1. \tag{28}$$

$$Q_0 = -\frac{A_1}{a} - \frac{A_2}{a^2}, \tag{29}$$

whence

$$A_2 = -a^2 Q_0 \tag{30}$$

but by (v), Art. **8**, and the critical relation, we have

$$a^2 = R^2/4L^2 = 1/LC.$$

Therefore

$$A_2 = -Q_0/LC = -E_0/L, \text{ provided } R^2 = 4L/C. \tag{31}$$

15. Complete Solution for Current Subject to the Initial Conditions.—Having determined the values of the arbitrary constants A_1 and A_2 subject to the initial conditions of the prob-

lem of the condenser discharge, let us now substitute their values in equations (20) and (21).

We obtain from (20), which is the value of i for the non-critical case, the result

$$i = -\frac{E_0}{2\omega_h L}\left\{\epsilon^{k_1 t} - \epsilon^{k_2 t}\right\}.$$

Replacing k_1 and k_2 by their values from (vi) and (vii), we obtain

$$i = -\frac{E_0}{L\omega_h}\epsilon^{-at}\left\{\frac{\epsilon^{\omega_h t} - \epsilon^{-\omega_h t}}{2}\right\}. \tag{32}$$

or, replacing ω_h by its equivalent value $j\omega$, this may be written in the alternative form

$$i = -\frac{E_0}{L\omega}\epsilon^{-at}\left\{\frac{\epsilon^{j\omega t} - \epsilon^{-j\omega t}}{2j}\right\}. \tag{33}$$

It is seen that (32) and (33) may be respectively written

$$i = -\frac{E_0}{L\omega_h}\epsilon^{-at}\sinh\omega_h t \tag{34}$$

and

$$i = -\frac{E_0}{L\omega}\epsilon^{-at}\sin\omega t. \tag{35}$$

Returning now to the value of i in the critical case, equation (21), and replacing the arbitrary constants by their values (28) and (31), we obtain

$$i = -\frac{E_0 t}{L}\epsilon^{-at}. \tag{36}$$

Equation (34) *or equation* (35) *gives the value of i in the non-critical case. Either of these equations may be used, but it is simpler to use* (34) *whenever ω_h is real (that is, when $R^2 > 4L/C$); and* (35) *whenever ω is real (that is, when $R^2 < 4L/C$).*

In the critical case (where $R^2 = 4L/C$) the solution is equation (36).

The values of a, ω_h and ω are given in Art. **8.**

16. Complete Solution of Quantity Subject to the Initial Conditions.—The value of q may be obtained by substitution of the values of the constants A_1 and A_2 into the equations for q (23) and (24), but we shall adopt the alternative method of integrating i with respect to time.

In the non-critical case, by taking the time integral of (35) we obtain

$$q = \int i\,dt$$
$$= -\frac{E_0}{L\omega}\int \epsilon^{-at} \sin \omega t\,dt$$
$$= \frac{E_0}{L\omega\sqrt{a^2 + \omega^2}}\,\epsilon^{-at} \sin \left\{ \omega t + \tan^{-1}\left(\frac{\omega}{a}\right)\right\}.$$

Therefore by (viii), Art. **8**,

$$q = \frac{E_0}{L\omega}\sqrt{LC}\;\epsilon^{-at} \sin \left\{ \omega t + \tan^{-1}\left(\frac{\omega}{a}\right)\right\}. \tag{37}$$

The corresponding integration of (34) gives

$$q = \frac{E_0}{L\omega_h}\sqrt{LC}\;\epsilon^{-at} \sinh \left\{ \omega_h t + \tanh^{-1}\left(\frac{\omega_h}{a}\right)\right\}. \tag{38}$$

In the critical case, in which $R^2 = 4L/C$, we may obtain q simply by substituting the values of A_1 and A_2 from (28) and (31) into (24), obtaining

$$q = \frac{E_0}{L}\left\{\frac{1}{a^2} + \frac{t}{a}\right\}\epsilon^{-at},$$

but by (v) and by the fact that in the critical case $R^2 = 4L/C$, we have

$$a^2 = R^2/4L^2 = 1/LC,$$

and therefore,

$$q = E_0C(1 + at)\,\epsilon^{-at}$$
$$= Q_0(1 + at)\,\epsilon^{-at}. \tag{39}$$

Equation (37) *or* (38) *gives the value of q in the non-critical case. Either of these equations may be used, but it is simpler and more direct to use* (38) *when ω_h is real* (*that is, when $R^2 > 4L/C$*) *and* (37) *when ω is real* (*that is, when $R^2 < 4L/C$*).

In the critical case (*where $R^2 = 4L/C$*) *the solution is* (39). *The values of a, ω_h and ω are given in Art.* **8.**

17. Identity of the Critical Case Results with the Non-critical. It is to be noted that, although the form of the expression derived for i and q in the critical case is different from the form obtained in the non-critical case, in reality the non-critical results reduce to the critical results if we make

$$R^2 = 4L/C.$$

This may be shown as follows: If

$$R^2 = 4L/C$$

equation (iv) gives

$$\omega = 0.$$

Now

$$\lim_{\omega \doteq 0} \left\{ \frac{\sin \omega t}{\omega} \right\} = \lim_{\omega \doteq 0} \left\{ \frac{\omega t - (\omega t)^3/3! + \cdots}{\omega} \right\} = t;$$

whence by (35) the current, as ω approaches 0, approaches

$$i = -\frac{E_0 t}{L} \epsilon^{-at} = -\frac{E_0 t}{L} \epsilon^{-\frac{R}{2L}};$$

which is the current in the critical case, as given in (36).

If next we concern ourselves with the limit approached by the charge q of equation (37) as ω approaches 0, and note that we may expand $\tan^{-1}(\omega/a)$ for small values of ω/a in the form

$$\tan^{-1}\left(\frac{\omega}{a}\right) = \frac{\omega}{a} - \frac{1}{3}\left(\frac{\omega}{a}\right)^3 + \ . \ . \ . \quad \text{[B. O. Peirce's Tables, No. 779]}$$

we find that

$$\lim_{\omega \doteq 0} \left\{ \frac{\sin\left[\omega t + \tan^{-1}\left(\frac{\omega}{a}\right)\right]}{\omega} \right\} = t + \frac{1}{a}$$

This substituted in (37) gives

$$q = \sqrt{LC}\, \frac{E_0(t + 1/a)}{L} \epsilon^{-at}$$

$$= \sqrt{LC}\, \frac{Q_0(t + 1/a)}{LC} \epsilon^{-at}.$$

Now by the fact that in the critical case

$$R^2 = 4L/C,$$

we have

$$\frac{1}{\sqrt{LC}} = a;$$

therefore

$$q = Q_0(1 + at)\epsilon^{-at},$$

which is in agreement with the value of q for the critical case, equation (39).

We thus obtain the result that after the determination of the constants of integration, the critical-case solution, although apparently very different in form from the non-critical case, is in reality

comprised in the non-critical solutions. We need thus, as the result of the discharge problem, only one equation for i (35) and one for q (37) whatever the value of R^2 in relation to $4L/C$. The other values of i and q given in (34) and (38) are more directly applicable when ω is imaginary; and those given in (36) and (39) are more directly applicable when ω is zero.

Before entering upon an examination of the results for the current and quantity during the discharge of a condenser, we shall first investigate the analogous problem for the charging of a condenser under the action of a constant impressed e.m.f.

II. THE CHARGING OF A CONDENSER

18. The Charging of a Condenser Under a Constant Impressed E.M.F.—Let

E = the constant impressed e.m.f.

The counter electromotive forces are the same as in the preceding problem, so that the differential equation for current is

$$E = L\frac{di}{dt} + Ri + \frac{\int i dt}{C}. \tag{40}$$

Differentiating (40) we obtain

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}. \tag{41}$$

To obtain the differential equation for q, we may substitute for i in (40) its value

$$i = \frac{dq}{dt},$$

obtaining

$$E = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \tag{42}$$

Equations (41) and (42) are the required differential equations. We shall not distinguish between the critical and the non-critical cases, but after the final solution has been obtained and the arbitrary constants determined, the critical case will appear as a special case of the non-critical, or general, case.

The complete solution of (41) has already been obtained in (20) in the form

$$i = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t}. \tag{43}$$

The complete solution of (42) is

$$q = B_1\epsilon^{k_1t} + B_2\epsilon^{k_2t} + CE, \tag{44}$$

by Appendix I, Note 8, or as follows:

The result (44) may be obtained by adding the *particular integral* of (42) (namely, CE) to the *complementary function* which is the solution of (42) with the constant E replaced by zero. This complementary function is

$$B_1\epsilon^{k_1t} + B_2\epsilon^{k_2t}.$$

Having now the values of i and q in (43) and (44), we are now to observe that to make i equal to the time derivative of q, we must require that

$$B_1 = A_1/k_1 \quad \text{and} \quad B_2 = A_2/k_2;$$

so that we may write q in the form

$$q = \frac{A_1}{k_1}\epsilon^{k_1t} + \frac{A_2}{k_2}\epsilon^{k_2t} + CE. \tag{45}$$

Let us now insert the initial conditions that the condenser shall start uncharged and that the initial current shall be zero; that is,

$$\text{when} \quad t = 0, \quad q = 0, \quad \text{and} \quad i = 0.$$

These conditions give

$$0 = A_1 + A_2,$$

and

$$0 = \frac{A_1}{k_1} + \frac{A_2}{k_2} + CE;$$

whence

$$A_2 = -A_1,$$

and

$$A_1 = CE\,\frac{k_1k_2}{k_1 - k_2}$$
$$= \frac{E}{2L\omega_h} \quad \text{by (viii) and (ix).}$$

Now by comparison it will be seen that A_1 and A_2 are the negatives of the values obtained for these quantities in equations (25) and (27) (which give the current and quantity during the discharge), except that the E which appears in the present problem is the e.m.f. impressed on the circuit, while in the discharge

problem E_0 is the potential difference to which the condenser was initially charged

In the event that the condenser is first charged under the impressed e.m.f. E and then discharged, these two values of E are the same. If t is measured from the beginning of the charging in the one case and from the beginning of the discharging in the other case, it will be seen that the current in the two cases differs only in sign, and that the quantity during charge is a constant CE minus the quantity during discharge.

Expressed mathematically, these results are contained in the following table:

19. Comparison of Discharge with Charge.—

During discharge	During charge
t = Time from beginning of discharge,	t = Time from beginning of charge,
E_0 = Difference of potential between the plates of the condenser at $t = 0$,	E = Difference of potential between the plates of the con denser at $t = \infty$,
Q_0 = Charge on positive plate when $t = 0$,	Q = Charge on positive plate when $t = \infty$,
i = Current toward the positive plate at time t,	i = Current toward the positive plate at time t,
q = Charge on the positive plate at time t.	q = Charge on the positive plate at time t.

then

$$i = -\frac{E_0}{L\omega}\epsilon^{-at}\sin\omega t, \tag{46}$$

$$q = \sqrt{LC}\frac{E_0}{L\omega}\epsilon^{-at}\sin\left(\omega t + \tan^{-1}\frac{\omega}{a}\right), \tag{47}$$

then

$$i = +\frac{E}{L\omega}\epsilon^{-at}\sin\omega t, \tag{48}$$

$$q = CE - \sqrt{LC}\frac{E}{L\omega}\epsilon^{-at}\sin\left(\omega t + \tan^{-1}\frac{\omega}{a}\right). \tag{49}$$

Note that in the case in which ω is not real, these quantities are the same as here given, but may be more conveniently used with hyperbolic sines and hyperbolic antitangents in place of sin and tan, and with ω_h substituted for ω.

Also the result for the critical case is comprised in the above equations. We may, however, simplify the result in the critical case, by taking the limits of i and q above as ω approaches zero. This process gives *for the critical case*

During discharge

$$i = -\frac{E_0 t}{L}\epsilon^{-at}, \tag{50}$$

$$q = Q_0(1 + at)\epsilon^{-at} = CE_0(1 + at)\epsilon^{-at}. \tag{51}$$

During charge

$$i = +\frac{Et}{L}\epsilon^{-at}, \tag{52}$$

$$q = CE - Q(1 + at)\epsilon^{-at} = CE - CE(1 + at)\epsilon^{-at}. \tag{53}$$

III. PERIOD, DAMPING FACTOR, AND LOGARITHMIC DECREMENT

20. Determination of Period During Discharge.—We come now to a discussion of the results obtained in the case of the condenser discharge. We have found for the current and quantity during discharge the equations

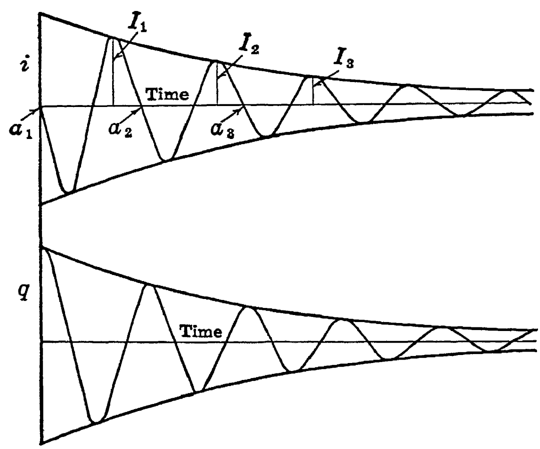
$$i = -\frac{E_0}{L\omega}\epsilon^{-at}\sin\omega t, \tag{54}$$

$$q = \sqrt{LC}\frac{E_0}{L\omega}\epsilon^{-at}\sin\left(\omega t + \tan^{-1}\frac{\omega}{a}\right), \tag{55}$$

in which i is the current flowing toward the plate that was initially positively charged, and q is the quantity of electricity on this plate at the time t.

FIG. 3.

FIG. 4.



FIGS. 3 and 4.—Giving respectively current i and quantity q (on positive plate of condenser) plotted against time.

If ω is real (that is, if $R^2 < 4L/C$) both of these quantities are seen to be periodic and to have a factor that is a sinusoidal function of the time.

A diagram of i plotted against t is given in Fig. 3. A similar diagram for q is given in Fig. 4.

The period of oscillation of the current in Fig. 3 may be defined as the time between alternate zero values of the current; that is, the time between the points a_1 and a_2, a_2 and a_3, etc. These points are the values of t for which i becomes zero after successive

complete cycles, and by (54) they occur at values of t for which

$$\sin \omega t = 0.$$

Since only alternative points are considered, this relation gives

$$\omega t = 0,\ 2\pi,\ 4\pi,\ \text{etc.};$$

whence, giving subscripts to different values of t satisfying the relation, we have

$$t_1 = 0,$$
$$t_2 = 2\pi/\omega,$$
$$t_3 = 4\pi/\omega,\ \text{etc.}$$

and, therefore, the period T is

$$T = t_2 - t_1 = t_3 - t_2 = \ .\ .\ .\ = 2\pi/\omega.$$

Putting in the value of ω from (iv), we obtain

$$T = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \qquad \text{(Thomson's Formula)}. \qquad (56)$$

Equation (56) gives T the period of oscillation of the current during the discharge of a condenser. Similar reasoning gives the same period of oscillation of the quantity q. It is seen that this period is real, only provided

$$R^2 \leqq 4L/C. \qquad (56a)$$

21. Approximate Value of the Period of Discharge.—Assuming that the inequality (56*a*) is satisfied, it is seen that

$$\frac{R^2}{4L^2} \leqq \frac{1}{LC},$$

then the equation (56) may be expanded by the binomial theorem into

$$T = 2\pi\sqrt{LC}/\{1 - \frac{LCa^2}{2} + \frac{3L^2C^2a^4}{8} - \ .\ .\ .\} \qquad (57)$$

where

$$a = \frac{R}{2L}.$$

Now if we note by (viii) that

$$\frac{1}{LC} = a^2 + \omega^2,$$

we shall see that equation (57) reduces to

$$T = 2\pi\sqrt{LC}, \tag{58}$$

(Thomson's approximation formula) provided

$$\frac{a^2}{2} < < \omega^2, \text{ or } \frac{R^2}{2} < < \frac{4L}{C} \tag{58a}$$

where the symbol $< <$ means "is negligible in comparison with."

The approximate period T calculated by the formula (58) has an important rôle in some of the work of the later chapters and is called the *Undamped Period of the Circuit* and will often be designated by S to distinguish it from the true period T.

The Equation (58) *gives the Undamped Period of the oscillations of current during the discharge of the condenser. This is sensibly the actual free period of the oscillations if $a^2/2$ is negligible in comparison with ω^2. The oscillations of q have the same period as the oscillations of i.*

22. The Time between Successive Positive or Negative Maxima is the Same as the Time between Alternate Zero Values.—Let us next find the time between successive positive or negative maxima of current and show that this gives the same period as the time between alternate zero values of current.

Equation (54) for the current is

$$i = -\frac{E_0}{L\omega}\epsilon^{-at}\sin\omega t.$$

Differentiating this with respect to the time and setting $di/dt = 0$, we have

$$0 = \frac{E_0\sqrt{\omega^2 + a^2}}{L\omega}\epsilon^{-at}\sin\left(\omega t + \tan^{-1}\frac{\omega}{-a}\right),$$

whence

$$\omega t = -\tan^{-1}\frac{\omega}{-a}, \ -\tan^{-1}\frac{\omega}{-a} + 2\pi, \ -\tan^{-1}\frac{\omega}{-a} + 4\pi, \ . \ . \ .$$

If we let the successive values of t obtained from this equation be t_1, t_2, t_3, etc. and let $\varphi = \tan^{-1}(\omega/-a)$, we have

$$\begin{aligned} t_1 &= -\varphi/\omega, \\ t_2 &= -\varphi/\omega + 2\pi/\omega, \\ t_3 &= -\varphi/\omega + 4\pi/\omega. \end{aligned} \tag{59}$$

Now the interval of time between the successive maxima is T' (say), and is seen to be

$$T' = t_2 - t_1 = t_3 - t_2 = \quad . \; . = 2\pi/\omega = T.$$

The time between successive maxima of the same sign is the same as the time between alternative zero values of current. This same fact is true in regard to quantity.

23. Period of Oscillation During Charging.—If R^2 is less than $4L/C$, the angular velocity ω that occurs in the equations for charge or discharge of the condenser is a real quantity and the charge or discharge is oscillatory. From the similarity of the equations for charge and discharge, it is seen that the period of oscillation of current or quantity during charge is the same as the period of oscillation during discharge for a circuit of given constants.

24. Logarithmic Decrement. Damping Constant.—In the equations given above we have seen that, in case R^2 is less than $4L/C$ the discharge of the condenser and the charge of the condenser are oscillatory, and we have determined the period of the oscillation, and have also proved that the period between maxima or minima is the same as the period between zero values. The oscillation is, however, not purely sinusiodal, because the equations for i and q involve an exponential factor with negative exponent. This exponential factor starts with the value 1 when $t = 0$, and decreases with increasing t, and becomes 0 when $T = \infty$; that is, the amplitude of the oscillations becomes smaller and smaller with increasing time. This process is called *damping* and the factor ϵ^{-at} is called the *damping factor.* The constant a is called the *damping constant.*

If we designate successive maxima of current in the same direction by I_1, I_2, I_3, etc., as indicated in Fig. 3, we shall have by equations (54) and (59)

$$I_1 = -\frac{E}{L\omega}\epsilon^{-at_1}\sin\varphi$$

$$I_2 = -\frac{E}{L\omega}\epsilon^{-a(t_1 + T)}\sin\{2\pi + \varphi\}$$

$$I_2 = -\frac{E}{L\omega}\epsilon^{-a(t_1 + 2T)}\sin\{4\pi + \varphi\},$$

etc.

Now in these several equations the sine terms are the same; therefore, by division, we obtain

$$I_1/I_2 = I_2/I_3 = \epsilon^{aT}:$$

whence

$$aT = \log_\epsilon I_1 - \log_\epsilon I_2 = \log_\epsilon I_2 - \log_\epsilon I_3, \ . \ . \ . \tag{61}$$

Let us designate aT by a single letter d, then

$$d = aT = \frac{RT}{2L}. \tag{62}$$

It is seen that the natural logarithm of the amplitude of the current falls by a constant amount d during each complete oscillation, or cycle, that is, d is the decrement per cycle of the logarithm of the amplitude of the current. This quantity d is called the logarithmic decrement per cycle, abbreviated Log. Dec. It is seen that the Log. Dec. of q is the same as that of i.

In terms of the logarithmic decrement d the equations (35) and (37) for current and quantity during the oscillatory discharge of the condenser may be written

$$i = -\frac{E_0}{L\omega}\epsilon^{-\frac{d\ t}{T}} \sin \omega t, \tag{63}$$

$$q = \sqrt{LC}\,\frac{E_0}{L\omega}\epsilon^{-\frac{d \cdot t}{T}} \sin\left(\omega t + \tan^{-1}\frac{T\omega}{d}\right). \tag{64}$$

IV. EXCITATION BY CURRENT INTERRUPTION

25. The Production of Oscillations by Buzzer Excitation.—In many of the experiments employed in high-frequency measurements electrical oscillations are produced by excitation of the condenser circuit by the use of a buzzer,[1] which acts by making and breaking a current flowing in an inductance.

The accompanying figure (Fig. 5) represents a battery B supplying current to an inductance L through an interrupter J. The inductance L has resistance R, and is shunted by a condenser of capacity C.

The interrupter J is here represented as a buzzer with its field coil L_0 also shunted by a condenser C_0.

The mathematical theory which follows applies to the heavy line circuit LRC, which is a circuit of frequency high in comparison with that of the circuit L_0C_0.

[1] This form of buzzer excitation is due to Zenneck, *Leitfaden der drahtlosen Telegraphie*, p. 3.

Let us measure time from the instant of interruption of current at J. Let the current flowing in L at any time t seconds after the interruption be i, which is a function of t, and let the charge in the condenser C at the same time be q.

Then

$$i = dq/dt. \tag{65}$$

From the time $t = 0$, when the circuit is broken at J, there is no external impressed e.m.f., so the differential equations for current and quantity are

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} \tag{66}$$

and

$$0 = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \tag{67}$$

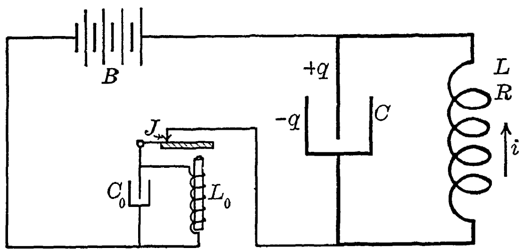


FIG 5 —Diagram of circuits for buzzer excitation.

The complete solution of (66) and (67) gives

$$\left.\begin{aligned} i &= A_1\,\epsilon^{k_1t} + A_2\,\epsilon^{k_2t}, \\ q &= \frac{A_1}{k_1}\,\epsilon^{k_1t} + \frac{A_2}{k_2}\,\epsilon^{k_2t}, \end{aligned}\right\} \tag{68}$$

where k_1 and k_2 have the values given in (i) and (ii), Art. 8.

The initial conditions are

$$\text{When } t = 0, \quad i = I, \tag{69}$$

and

$$\text{when } t = 0, \quad q = -CRI, \tag{70}$$

where I is the current flowing in the coil L at the time of interruption at J. Equation (70) is obtained on the assumption that the current is in practically steady state immediately before interruption, so that the counter e.m.f. in the coil is RI. This is the potential of the lower plate of the condenser in excess of the

upper plate. The capacity C times this potential gives the charge on the lower plate as CRI; but the upper plate is regarded as positive, whence the negative sign in (70). The charge is $-CRI$.

If now we introduce the initial conditions (69) and (70) into the pair of equations (68), we obtain

$$I = A_1 + A_2, \tag{71}$$

$$-CRI = \frac{A_1}{k_1} + \frac{A_2}{k_2} = \frac{k_2 A_1 + k_1 A_2}{k_1 k_2}. \tag{72}$$

A determination of the A's from these equations may be made as follows:

From (72), by the relations (v), (vii), and (viii), we obtain

$$-CRI = LC\{(A_1 + A_2)\ (-a) + j\omega(A_2 - A_1)\}.$$

$$\text{Now } a = R/2L,$$

whence

$$I = \frac{A_1 + A_2}{2} + \frac{j\omega(A_1 - A_2)}{2a}.$$

This equation, combined with (71), gives

$$A_1 - A_2 = aI/j\omega,$$

which combined with (71) gives

$$A_1 = \frac{aI}{2j\omega} + \frac{I}{2},$$

$$A_2 = -\frac{aI}{2j\omega} + \frac{I}{2},$$

Substitution of these values of A_1 and A_2 into the equation (68) for i gives, in view of (vi) and (vii),

$$\begin{aligned} i &= \epsilon^{-at}\{A_1\epsilon^{j\omega t} + A_2\epsilon^{-j\omega t}\} \qquad (73) \\ &= \epsilon^{-at}\left\{\frac{I}{2}\left(\epsilon^{j\omega t} + \epsilon^{-j\omega t}\right) + \frac{aI}{2j\omega}\left(\epsilon^{j\omega t} - \epsilon^{-j\omega t}\right)\right\} \\ &= \frac{I}{\omega}\epsilon^{-at}(\omega \cos \omega t + a \sin \omega t) \\ &= \frac{I\sqrt{a^2 + \omega^2}}{\omega}\epsilon^{-at} \sin\left(\omega t + \tan^{-1}\frac{\omega}{a}\right). \end{aligned}$$

By (viii) this last equation gives

$$i = \frac{I}{\omega\sqrt{LC}}\epsilon^{-at} \sin\left(\omega t + \tan^{-1}\frac{\omega}{a}\right). \tag{74}$$

Equation (74) gives the current in the coil L in the direction of the original current I, at a time t seconds after the interruption of I at J.

26. On the E.M.F. Induced in a Very Loosely Coupled Secondary Circuit by Buzzer Excitation.—In the preceding sections there has been discussed the oscillations produced in a circuit by a method known as buzzer excitation. Oscillations produced in this way are often employed to impress an e.m.f. on a secondary circuit for the purpose of making measurements in the secondary circuit. A diagram of this arrangement of apparatus is shown in Fig. 6.

The oscillations occurring in the circuit LC impress an e.m.f. on the circuit L_2C_2. Let us now specify that the circuit L_2C_2,

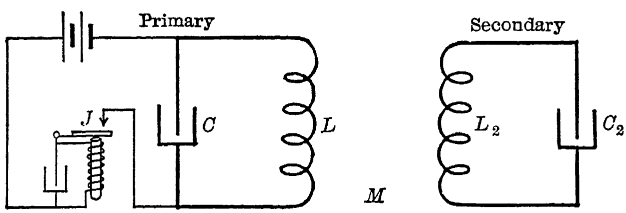


FIG. 6—Buzzer excitation of primary circuit inducing e m f in secondary circuit

which we shall call the secondary circuit, shall be so far away from the primary circuit LC that the current induced in the secondary does not materially influence the current flowing in the primary, and let us determine the e.m.f. induced in the secondary circuit.

The induced e.m.f. has the instantaneous value determined by the mutual inductance M between the two circuits and by the time rate of change of the primary current. The relation is

$$e_2 = M\frac{di}{dt}. \tag{75}$$

Substituting in (75) the value of i from (74) and performing the differentiation, we have

$$e_2 = \frac{MI}{\omega\sqrt{LC}}\,\epsilon^{-at}\left\{-a\sin(\omega t+\varphi)+\omega\cos(\omega t+\varphi)\right\},$$

where

$$\varphi = \tan^{-1}\frac{\omega}{a}.$$

Expanding the sine and cosine terms by the formulas for sines and cosines of a sum and noting that

$$\sin \varphi = \omega/\sqrt{\omega^2 + a^2} = \omega\sqrt{LC}, \text{ by (viii), and}$$
$$\cos \varphi = a\sqrt{LC},$$

we have

$$e_2 = -\frac{MI}{LC\omega}\epsilon^{-at}\sin \omega t. \tag{76}$$

Equation (76) *gives the instantaneous value of the e.m.f. impressed on a loosely coupled secondary circuit by buzzer excitation of a primary circuit.*

Let us suppose now that we are to impress different frequencies of e.m.f. on the secondary circuit by giving C in the primary various values, and see how the impressed e.m.f. depends on the frequency. We shall get the result only approximately, by supposing that the decrement of the primary current is so small that a^2 is negligible in comparison with ω^2, then by (viii)

$$\omega^2 = 1/LC \text{ approximately,}$$

and (76) becomes

$$e_2 = -M\omega I\epsilon^{-at}\sin \omega t, \text{ approximately.} \tag{77}$$

Equation (77) *gives an approximate value of the instantaneous value of the e.m.f. impressed on a loosely coupled secondary circuit by a primary circuit excited by a buzzer and varied as to frequency by varying the condenser in the primary circuit. The induced e.m.f. is in this case proportional to the frequency of the primary circuit—this frequency being $n = \omega/2\pi$.*

CHAPTER III

ENERGY TRANSFORMATIONS DURING CHARGE OR DISCHARGE OF A CONDENSER

27. Notation.—

p = Instantaneous value of power,
P = Average power,
W = Energy stored,
W_R = Energy expended in a resistance,
W_{12} = Energy supplied during time from t_1 to t_2,
W_t = Energy in system at time t,
$\bar{I}^2$ = Mean-square current,
$\bar{I}$ = Square root of mean-square current (R.M.S. current).

28. General Notions Regarding Power and Energy.—Let us suppose that we have two conducting terminals A and B protruding through the side of a room, and that we do not know what kind of electrical circuit or electrical apparatus is within the room, except that it is such that when we connect the terminals A and B to a given electrical system outside of the room, the current at any instant flowing out at B has the same magnitude as the current flowing in at A.

Then if

i = the instantaneous value of current flowing into the room at A, and
e = the instantaneous excess of potential at A over that at B,

the instantaneous power p flowing into the room, or supplied from without to the apparatus within the room, is

$$p = ei. \tag{1}$$

This equation is based immediately upon fundamental definitions; for the excess of potential e of A over B is, by definition of potential, the work that must be expended by an outside system to send a unit quantity of electricity from A to B. To

send i units of electricity per second requires an amount of work per second, or power, equal to $e \times i$.

Returning to the power equation (1) let us note that either e or i, or both of them may be negative. Keeping the definitions of i and e given above and merely interchanging the letters A and B attached to the terminals will change the sign of both e and i, but will, therefore, not affect the value of the product p. If on the other hand, the disposition of the apparatus within the room is such that current comes out at A, while A has a higher potential than B, then the instantaneous power is negative, and the apparatus within the room is at the given instant supplying power to the apparatus outside of the room.

The energy supplied to the apparatus within the room from without, during a time extending from t_1 to t_2, is the time integral of the power; that is

$$W_{12} = \int_{t_1}^{t_2} p dt. \tag{2}$$

29. Power Supplied to a Perfect Condenser.—Suppose that the two terminals A and B that were thought of as protruding from a room, are the terminals of a condenser of capacity C. The condenser will be defined as *perfect* if it is such that C is constant and independent of q in the equation

$$q = Ce, \tag{3}$$

where

q = the instantaneous charge upon that plate of the condenser that is attached to the terminal A,

e = the excess of potential of A over B,

and

C = the capacity of the condenser.

The relations dealt with in the previous chapters assumed that the condensers employed were perfect condensers. Unless otherwise stated the condensers throughout the book will be assumed to be perfect.

In the case of the condenser attached to the terminals A and B let

i = the instantaneous current flowing in at A,

then

$$i = dq/di. \tag{4}$$

Combining equations (1), (3), and (4), we obtain for the instantaneous power p supplied to the condenser the value

$$p = ei = \frac{q}{C}\frac{dq}{dt} = \frac{1}{2C}\frac{d(q^2)}{dt} = \frac{C}{2}\frac{d(e^2)}{dt}. \tag{5}$$

Equation (5) gives the instantaneous value of the power p supplied to the perfect condenser of capacity C.

30. Energy Supplied to a Perfect Condenser.—The energy W_{12} supplied to the condenser from without in the interval of time from t_1 to t_2 is, by (2) and (5)

$$W_{12} = \int_{t_1}^{t_2} \frac{1}{2C}\frac{d(q^2)}{dt}\,dt, \text{ or } \int_{t_1}^{t_2} \frac{C}{2}\frac{d(e^2)}{dt}\,dt.$$

Integrating, we have

$$W_{12} = \frac{Q_2^2 - Q_1^2}{2C},$$

$$= \frac{C(E_2^2 - E_1^2)}{2}, \tag{6}$$

in which

E_2 and Q_2 = difference of potential and charge respectively at the time t_2,
E_1 and Q_1 = these quantities at the time t_1.

It is seen that the energy W_{12} supplied to the condenser depends only upon the initial and final state of the charge of the condenser, and is independent of the time required to effect the modification of the charge.

Equation (6) gives the energy that must be supplied to the condenser to raise the charge from Q_1 to Q_2 (or its potential from E_1 to E_2).

It is to be noted that if $Q_2 = \pm Q_1$, $W_{12} = 0$; that is, during any process in which the charge of the condenser is taken from a given value through any modification and brought back to the initial value, or its negative, the energy supplied to the condenser from without is zero.

Whatever energy is supplied to increase the charge of the condenser is stored in the static field, and is completely recovered when the condenser is brought to its initial state of charge, or to an equal charge of opposite sign.

If the condenser is initially uncharged, $Q_1 = 0$, and by equation (6)

$$W = \frac{Q^2}{2C} = \frac{CE^2}{2}. \tag{7}$$

Equation (7) gives the energy W required to charge a condenser of capacity C from an initially uncharged state to a final charge Q, with final potential difference E.

31. Power and Energy Supplied to a Resistanceless Inductance.—If the two terminals A and B in the preceding illustration are the terminals of a resistanceless inductance, and if

$$i = \text{the current flowing in at } A,$$

then the counter e.m.f. of the inductance will be

$$e = L\frac{di}{dt}.$$

This will be the excess of potential of A above that of B, and the power supplied to the inductance will be, by (1)

$$p = Li\frac{di}{dt} = \frac{L}{2}\frac{d(i^2)}{dt}. \tag{8}$$

The energy W_{12} supplied to the inductance during the time from t_1 to t_2 is seen by (2) and (8) to be

$$W_{12} = \int_{t_1}^{t_2}\frac{L}{2}\frac{d(i^2)}{dt}\,dt = \frac{1}{2}L(I_2^2 - I_1^2), \tag{9}$$

where

I_2 = current flowing in the inductance at the time t_2,
I_1 = corresponding current at the time t_1.

Equation (8) gives the power p supplied to the inductance at the time t.

Equation (9) gives the energy W_{12} supplied to the inductance L to change the current in the inductance from I_1 to I_2.

It is to be noted that if $I_2 = \pm I_1$, $W_{12} = 0$; that is, in any process during which the current starts at the value I_1, goes through any changes whatever and returns to I_1 or to $-I_1$, the total energy supplied to the resistanceless inductance is zero. Whatever energy is supplied to it while the absolute value of i is increasing is stored in the magnetic field and is recovered when the absolute value of i is again reduced by an equal amount.

If the initial current flowing in the inductance is $I_1 = 0$,

and the final current is $I_2 = I$, equation (9) gives for the energy stored in the inductance the value

$$W = \frac{1}{2}LI^2. \tag{9a}$$

Equation (9*a*) *gives the value of the energy stored in the inductance L when traversed by a current I.*

32. Power and Energy Supplied to a Resistance.—When a current i flows through a resistance, the counter e.m f. of the resistance is Ri, so that the power supplied to the resistance at any instant is

$$p = Ri^2. \tag{10}$$

The energy supplied during a time from t_1 to t_2 is

$$W_{12} = R\int_{t_1}^{t_2} i^2dt. \tag{11}$$

Equations (10) *and* (11) *give respectively the instantaneous power p, and the energy W_{12} supplied from t_1 to t_2, to a resistance R.*

It is to be noted that whether i is positive, negative, increasing, diminishing or steady, i^2 is positive and every increment of time during which current is flowing in the resistance adds to the energy expenditure.

33. Logarithmic Decrement of Energy of the Circuit During the Discharge of a Condenser.—Let us consider that a condenser of capacity C has been charged to an initial potential difference E and is allowed to discharge through a resistance and inductance, and let us determine the energy resident in the inductance and capacity at any time t seconds after the beginning of the discharge.

In general, at the time t, the condenser has a charge q given by equation (47), Art. **19**, and there is flowing through the inductance a current i given by (46), Art. **19**.

The total energy resident in the system is

$$W = \frac{q^2}{2C} + \frac{Li^2}{2}, \tag{12}$$

If we replace q and i by their values from (46) and (47), Art. **19**, we obtain

$$\begin{aligned} W &= \frac{LC}{2C}\frac{E^2}{L^2\omega^2}\epsilon^{-2at}\sin^2\left(\omega t + \tan^{-1}\frac{\omega}{a}\right) \\ &\qquad + \frac{L}{2}\frac{E^2}{L^2\omega^2}\epsilon^{-2at}\sin^2\omega t, \\ &= \frac{E^2}{2L\omega^2}\epsilon^{-2at}\left\{\sin^2\left(\omega t + \tan^{-1}\frac{\omega}{a}\right) + \sin^2\omega t\right\}. \end{aligned} \tag{13}$$

This is the electrical energy resident in the system at any time t. If now we take any two times t and $t + T$, where T is one whole period of oscillation, the sine terms will be identical at t and $t + T$, and we shall have as the ratio of the energies in the system at the two times

$$\frac{W_t}{W_{t+T}} = \frac{\epsilon^{-2at}}{\epsilon^{-2a(t+T)}} = \epsilon^{2aT},$$

whence

$$\log W_t - \log W_{t+T} = 2aT = 2d, \tag{14}$$

where

$$d = aT,$$

and is the logarithmic decrement of current or quantity per cycle.

Equation (14) *gives the logarithmic decrement of electrical energy in a condenser and inductance during discharge, and shows by comparison with* (61) *of* Art. **24** *that the log. dec. of energy per cycle is twice the log. dec. of current or quantity per cycle.*

34. Energy Expended in Resistance During Condenser Discharge.—In the preceding paragraph, we have examined the energy resident in the condenser and inductance during the discharge of a condenser. We shall now attack the complementary problem of determining how much energy has been dissipated in the resistance from the [illegible] of the discharge to a time t seconds thereafter.

If the time extends from $t = 0$ to $t = t$, the expended energy by (11) is

$$W_R = R\int_0^t i^2 dt. \tag{15}$$

Letting the initial value of condenser potential be E, we have, equation (46), Art. **19**,

$$i^2 = \frac{E^2}{L^2\omega^2}\epsilon^{-2at}\sin^2\omega t$$

Substituting this value of i^2 into (15) we obtain

$$\begin{aligned} W_R &= \frac{RE^2}{L^2\omega^2}\int_0^t \epsilon^{-2at}\sin^2\omega t\,dt \\ &= \frac{RE^2}{L^2\omega^2}\int_0^t \epsilon^{-2at}\frac{1 - \cos 2\omega t}{2}\,dt \\ &= \frac{RE^2}{L^2\omega^2}\left[\epsilon^{-2at}\left\{\frac{1}{-4a} - \frac{1}{4\sqrt{a^2+\omega^2}}\cos\left(2\omega t - \tan^{-1}\frac{\omega}{-a}\right)\right\}\right]_0^t \end{aligned} \tag{16}$$

This expression can be simplified by noting the trigonometric relation

$$\cos(x + y)\cos(x - y) = 1 - \sin^2 x - \sin^2 y,$$

whence

$$\cos(x + y) = \frac{1 - \sin^2 x - \sin^2 y}{\cos(x - y)} \tag{16a}$$

Now

$$-\cos\left(2\omega t - \tan^{-1}\frac{\omega}{-a}\right) = \cos\left(2\omega t + \tan^{-1}\frac{\omega}{a}\right) \tag{17}$$

If now we let

$$x = \omega t + \tan^{-1}(\omega/a)$$
$$y = \omega t,$$

and employ (16a), equation (17) becomes

$$-\cos\left(2\omega t - \tan^{-1}\frac{\omega}{-a}\right) = \frac{\sqrt{a^2 + \omega^2}}{a}\left[1 - \sin^2\left(\omega t + \tan^{-1}\frac{\omega}{a}\right) - \sin^2\omega t\right] \tag{18}$$

This result introduced into (16) gives, on replacing $a^2 + \omega^2$ and a by their values from (viii) and (v), Art. **8**,

$$W_R = \frac{E^2}{2L\omega^2}\left[\left\{-\sin^2\left(\omega t + \tan^{-1}\frac{\omega}{a}\right) - \sin^2\omega t\right\}\epsilon^{-2at}\right]_0^t$$
$$= \frac{E^2C}{2} - \left\{\frac{E^2}{2L\omega^2}\left[\sin^2\left(\omega t + \tan^{-1}\frac{\omega}{a}\right) + \sin^2\omega t\right]\right\}\epsilon^{-2at}. \tag{19}$$

Equation (19) *gives the energy W_R expended in the resistance R during the discharge of the condenser in the interval of time from the beginning of the discharge to t seconds thereafter,—the condenser being originally charged to a potential difference E.*

It is to be noticed that this energy expended in the resistance + the energy left in the circuit (13) *is $E^2C/2$, which is the energy originally in the condenser.*

It is also to be noticed that if we make t infinite, the terms involving t in (19) *become zero, and the total energy expended in the resistance becomes*

$$W_R = E^2C/2, \tag{20}$$

so that the energy lost in the resistance is the total energy of the condenser charge.

35. Average Current and Mean-square Current During N Complete Condenser Discharges per Second.—If we suppose that the condenser is charged N times per second, and after each

charging, the charging source is removed and the condenser is discharged through a current-measuring instrument whose deflections are proportional to the average current, we should have a measure of the average current of the N discharges per second.

If we assume that each discharge is practically complete, we can easily calculate this average current from fundamental considerations, as follows.

The quantity of electricity flowing from the condenser at each discharge is its original charge $= CE$. Per second the quantity is N times this, so that

The average current for N complete discharges per second

$$= NCE = NQ. \tag{21}$$

On the other hand, certain types of current-measuring instruments read the square root of the mean-square current (R.M.S. current). This is true of hot-wire ammeters, thermal-junctions, dynamometers, etc.

We shall, therefore, calculate from elementary considerations the R.M.S. current during N complete condenser discharges per second.

If there are N discharges per second, the energy dissipated in the resistance R of the circuit per second (that is, the average power P dissipated) is by (20).

$$P = NE^2C/2. \tag{22}$$

This average power divided by R gives the mean-square current, hence

$$\overline{I^2} = \frac{NE^2C}{2R}; \tag{23}$$

where I^2 with a dash over it means the mean-square current.

Taking the square root of the mean-square current (23), we obtain

R.M.S. current for N complete discharges per second

$$= \bar{I} = E\sqrt{\frac{NC}{2R}}. \tag{24}$$

Equations (21) *and* (24) *give respectively the mean current and the R.M.S. current obtained from N condenser discharges per second. The condenser is charged each time to a potential difference E, the charging source is removed, and the condenser is then*

discharged through any inductance L and resistance R. The inductance of the circuit is found to be immaterial, provided the discharge is complete.

In the case of the average current, both inductance and resistance are immaterial. The number of discharges N per second is supposed to be sufficiently small to permit practically complete discharges.

36. Energy Lost in the Resistance of the Circuit During the Charging of a Condenser.—We shall next prove a very interesting fact concerning loss of energy when a condenser is charged by applying a constant e.m.f. E.

During the process of charging a condenser through any resistance and inductance under the action of a constant impressed e.m.f E, the energy lost in the resistance of the circuit from time 0 to t is

$$W_R = R\int_0^t i^2 dt, \tag{25}$$

where t is measured from the beginning of the charge.

It is to be noticed that i^2 during the charge is of the same form as i^2 during discharge (equations (48) and (46), Art **19**) so that (25) when integrated gives the same result as (19), and when t is made infinite (see (20))

$$W_R = \frac{E^2C}{2} \tag{26}$$

Equation (26) gives the energy dissipated in the resistance of the circuit when a condenser C is charged by the application of a constant e.m.f. E. This amount of energy dissipated is independent of the inductance and resistance through which the condenser is charged. This energy dissipated is equal in amount to the energy finally delivered to the condenser, equation (7), so that the efficiency of the process of charging a condenser from a constant e.m f. applied through any inductance and resistance to the condenser is $\frac{1}{2}$; which means that in order to deliver any given amount of energy to a condenser by applying a constant e.m.f. an equal amount of energy must be dissipated in the resistance of the circuit, however small we make that resistance.

37. Energy and Power Supplied to a Condenser Circuit Excited by Current Interruption.—Reference is made to Fig. 5, Chapter II, which shows a circuit LRC excited by sending a practically steady current I through L and interrupting the current in the feed line.

After each interruption of the feed circuit at J, if the oscillations in the LRC circuit have time to die practically to zero before a new make of the interrupter, the energy expended in the resistance R is

$$W = \frac{LI^2}{2} + \frac{CR^2I^2}{2}, \tag{27}$$

as may be seen from the principle of the conservation of energy, since the first of these terms is the energy in the inductance and the second is the energy in the condenser at the beginning of the discharge.

If there are N makes and breaks of current at J each second, the energy per second (mean power P) dissipated in this circuit is

$$P = N\left(\frac{LI^2 + CR^2I^2}{2}\right). \tag{28}$$

Equation (28) *gives the average power P delivered to the oscillatory circuit LRC and expended in that circuit, provided the circuit is actuated by making and breaking a current I, N times per second, at J (Fig.* 5, *Chapter II), and provided the interruptions are sufficiently infrequent to allow a practically complete discharge of the inductance between interruptions, and provided the feed current I has time to come to a steady state in L.*

CHAPTER IV

THE GEOMETRY OF COMPLEX QUANTITIES

38. Utility.—In the mathematical treatment of periodic phenomena a considerable simplification is made by the use of imaginary and complex quantities. As aids to the memory, the complex quantities may be represented geometrically by simple diagrams, which are easier to remember than the algebraic formulas. By the use of a simple set of rules for the geometrical representation of algebraic quantities and algebraic operations (rules due to Argand and Demoivre) many of the algebraic manipulations may be performed by the aid of geometrical constructions; and the final results obtained may be reinterpreted, if necessary, into algebraic symbols for the purposes of calculations.

39. Representation of Real Quantities.—Real quantities are represented along a horizontal axis. This axis is called *the axis of reals.* As in analytical geometry, the numerical magnitudes of the real quantities are represented by lengths proportional to these magnitudes. Positive values of real quantities are represented by lengths drawn to the right along the axis of reals, from some arbitrary origin; negative values are represented by lengths drawn to the left from the origin.

A negative quantity may be looked upon as making an angle of 180°, or $180° + n\,360°$ with the positive axis of reals; while a positive quantity makes an angle $0° + n\,360°$ with this axis, where n is an integer.

Let us examine the result obtained by multiplying $+a$ by $-b$. The result is $-ab$, a quantity having a magnitude equal to the product of the magnitude of the factors, and an angle (180°) equal to the sum of the angle of the factors.

Likewise, the product of $-a$ by $-b$ is $+ab$, a quantity as before having a magnitude equal to the product of the magnitudes of the factors and an angle (360°) equal to the sum of the angles of the factors, since a line making an angle of 360° with the positive axis is coincident with a line making 0° with this axis.

As a third example, the multiplication of a quantity a by -1

reverses it, and a double multiplication of a by -1 is equivalent to a double reversal, or rotation through $180° + 180°$.

40. Representation of Imaginary Quantities. Argand's Method.—The quantity $\sqrt{-1}$ is a number that multiplied by itself gives -1. Also the double application of $\sqrt{-1}$ to a quantity a as a multiplier gives $-a$: this result is equivalent to the result obtained by rotating a through an angle of 180°. Consistent with this and with the fact that with real quantities double multiplication resulted in the addition of angles, let us postulate that the single operation of multiplying by $\sqrt{-1}$ amounts to a changing of a into a position it would have if rotated through 90°. That is, we shall represent geometrically $\sqrt{-1} \times a$ by a length a along an axis perpendicular to the axis of reals.

This vertical axis is called *the axis of imaginaries.* The $+$ and $-$ sign before imaginary quantities, as before real quantities, shows opposition in direction; that is, $+\sqrt{-1}\,a$ and $-\sqrt{-1}\,a$ have opposite directions along the axis of imaginaries as shown in Fig. 1. A detailed consideration of this method of representing real and imaginary quantities along two mutually perpendicular axes in the same plane shows that the system is entirely self-consistent. In order to avoid repeatedly writing $\sqrt{-1}$, we shall follow the prevailing custom in electrical engineering and adopt the symbol j for this quantity; that is

$$j = \sqrt{-1} \tag{1}$$
$$j^2 = -1.$$

41. Representation of Complex Quantities.—The complex quantity $a + bj$ shall be represented by the directed sect, or vector, OP, with a component a along the axis of reals and a component bj along the axis of imaginaries, Fig. 2. The directed sect, or vector, OP may be called the vectorial representation of the complex quantity, or briefly the vector OP may be called the vector $a + bj$.

A vector has magnitude and direction. The magnitude of the vector OP is the length of OP, which is

$$\sqrt{a^2 + b^2} = r \text{ (say)}. \tag{2}$$

The direction of OP is determined by the angle φ, which has the value

$$\varphi = \tan^{-1} \frac{b}{a}. \tag{3}$$

The polar coördinates of the point P are r and φ; and we may also describe the vector OP, or the complex quantity $a + bj$, which it represents, by a function of the coordinates r and φ. We shall now find two different expressions for this function,—one in trigonometric form, and the other in exponential form.

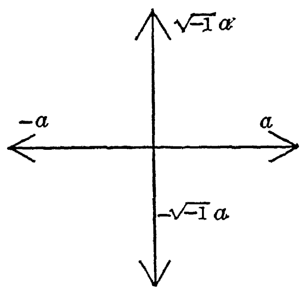


FIG. 1.—Representation of real and imaginary quantities

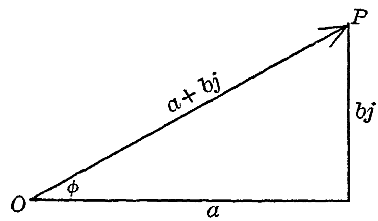


FIG 2—Representation of a complex quantity $a + bj$.

42. Trigonometric Expression for a Vector.—Let us take the complex quantity $a + bj$, and express it in terms of r and φ, where

$$r = \sqrt{a^2 + b^2}, \tag{4}$$

$$\varphi = \tan^{-1}\frac{b}{a} \tag{5}$$

From (5)

$$\tan\varphi = b/a, \tag{6}$$

$$\sin\varphi = b/r, \tag{7}$$

$$\cos\varphi = a/r. \tag{8}$$

If we multiply and divide $a + bj$ by r, we have the identity

$$a + bj = r(\cos\varphi + j\sin\varphi). \tag{9}$$

The function $r(\cos\varphi + j\sin\varphi)$ *is the trigonometric polar coördinate expression for the complex quantity* $a + bj$, *or for the vector* OP, *Fig.* 2.

43. Exponential Expression for the Vector OP. Demoivre's Formula.—Another form of expression for the vector OP in polar coördinates may be obtained by examining the series expansions of $\cos\varphi$, $\sin\varphi$, and $\epsilon^{j\varphi}$; to wit

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \tag{10}$$

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \ . \ . \ . \tag{11}$$

$$\epsilon^{j\varphi} = 1 + \frac{j\varphi}{1} + \frac{(j\varphi)^2}{2!} + \frac{(j\varphi)^3}{3!} + \ . \ . \ . \tag{12}$$

By combining these quantities we have

$$\cos\varphi + j\sin\varphi = \epsilon^{j\varphi}; \tag{13}$$

whence by equations (9) and (13)

$$a + bj = r\epsilon^{j\varphi}, \tag{14}$$

in which r is the length of the vector $a + bj$, and φ is the angle of the vector *expressed in radians.*

Equation (14) *gives a very convenient polar coördinate expression for a vector of length* r *making an angle* φ *radians with the axis of reals, Fig.* 3.

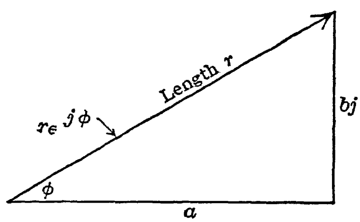


FIG. 3 —Polar-coördinate representation of $r\epsilon^{j\phi}$.

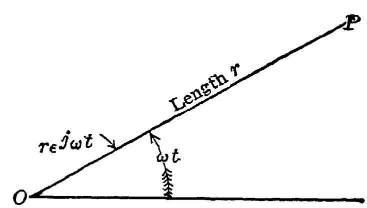


FIG 4 —Uniform circular motion

44. Exponential Expression for a Uniform Circular Motion.— If φ is the angle of the vector $r\epsilon^{j\varphi}$, and if this angle is made to vary uniformly with the time, we may write

$$\varphi = \omega t, \tag{15}$$

where ω is a constant.

Under these conditions the vector, Fig. 4, indicated by OP revolves around the point O with uniform velocity ω in a positive direction, as indicated by the arrow. The function

$$r\epsilon^{j\varphi} = r\epsilon^{j\omega t} \tag{16}$$

therefore represents a uniform circular motion in which the angle ω radians is described in a unit time.

The angle ω *described per unit time is called the angular velocity of the revolution.*

For the radius OP to move through an angle 2π radians (*i.e.*, once around) requires a time T such that

$$2\pi = \omega T, \tag{17}$$

or

$$T = 2\pi/\omega. \tag{18}$$

T given by (18) *is the period of revolution.*

45. The Addition of Complex Quantities, and the Summation of Vectors.—Returning now to general elementary considerations, let us suppose that we have two complex quantities

$$\left.\begin{aligned} z_1 &= a_1 + b_1 j \\ \text{and} \qquad z_2 &= a_2 + b_2 j. \end{aligned}\right\} \tag{19}$$

By direct algebraic addition their sum z is seen to be

$$z = z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)j. \tag{20}$$

From this it is seen that the geometrical representation of z, which is the sum of the complex quantities z_1 and z_2, is ob-

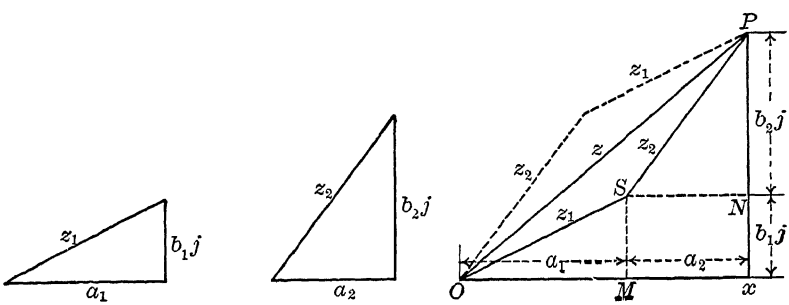


FIG. 5.—Addition of vectors

tained by laying off $a_1 + a_2$ on the axis of reals, giving the point x, Fig. 5, then at x a length $b_1 + b_2$ must be laid off in the direction of the axis of imaginaries. This brings us to the point P. The vector OP is z, the sum of z_1 and z_2.

If now through the points M and N respectively we draw the vertical line MS and the horizontal line NS, and join the intersection point S with O and P, we see that OS and SP are in magnitude and direction equivalent to z_1 and z_2 respectively.

Therefore, the geometrical sum of two vectors z_1 and z_2 is obtained by putting z_2 on the end of z_1, and joining the beginning

of z_1 with the end of z_2. The same result is obtained if z_1 is put on end of z_2, as shown by dotted lines in Fig. 5. The sum is again the vector OP and is now obtained by joining the beginning of the dotted z_1 to the end of the dotted z_2.

In like manner, the vector z is the sum of the vectors z_1, z_2, z_3, z_4, z_5, in Fig. 6. The vector sum of $z_1, z_2, \ldots z_5$ is independent of the order of the addition of terms. For example, if the order z_1, z_5, z_2, z_3, z_4 be taken the construction in Fig. 7 is obtained, which has the same sum as that obtained in Fig. 6.

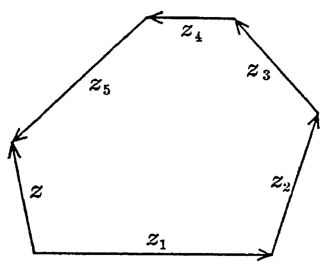


FIG. 6.—Addition of five vectors.

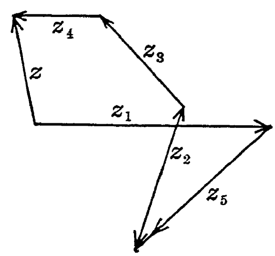


FIG. 7.—Addition of same vectors in different order.

46. The Multiplication of Complex Quantities, and the Geometrical Representation of the Product.—Given

$$\begin{aligned} z_1 &= a_1 + b_1 j \\ z_2 &= a_2 + b_2 j. \end{aligned} \tag{21}$$

Let

$$\begin{aligned} r_1 &= \sqrt{a_1^2 + b_1^2}, \\ r_2 &= \sqrt{a_2^2 + b_2^2}, \end{aligned} \tag{22}$$

$$\begin{aligned} \varphi_1 &= \tan^{-1} \frac{b_1}{a_1}, \\ \varphi_2 &= \tan^{-1} \frac{b_2}{a_2}. \end{aligned} \tag{23}$$

Let it be required to find the product of z_1 and z_2. We shall annunciate the rule in advance of proof.

Rule.—The product of two complex quantities z_1 and z_2 (as we shall immediately prove) is a new complex quantity represented by a length $r_1 r_2$ and making an angle $\varphi_1 + \varphi_2$ with the positive axis of reals. That is, the result of multiplying together of two complex quantities is obtained by multiplying their magnitudes and adding their angles.

Two proofs of this proposition follow:

Exponential Proof.—Put z_1 and z_2 into their exponential forms

$$z_1 = r_1\epsilon^{j\varphi_1},$$
$$z_2 = r_2\epsilon^{j\varphi_2};$$

whence by direct multiplication

$$z_1z_2 = r_1r_2\epsilon^{j(\varphi_1 + \varphi_2)}. \tag{24}$$

The product has, therefore, the product of r_1 and r_2 for its magnitude, and the sum of the angles φ_1 and φ_2 for its angle.

Alternative Proof.—The convenience of the use of the exponential function in operations involving multiplication is evident from the preceding paragraph. Let us compare with it the more involved process of direct multiplication of the algebraic form of the complex quantities.

Writing

$$z_1 = a_1 + b_1j,$$
$$z_2 = a_2 + b_2j,$$

and taking the product, we have

$$z_1z_2 = a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)j. \tag{25}$$

Now as in equations (7) and (8)

$$a_1 = r_1 \cos \varphi_1, \qquad b_1 = r_1 \sin \varphi_1.$$
$$a_2 = r_2 \cos \varphi_2, \qquad b_2 = r_2 \sin \varphi_2.$$

These values introduced into equation (25) give

$$z_1z_2 = r_1r_2\{(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + j(\cos \varphi_1 \sin \varphi_2 + \cos \varphi_2 \sin \varphi_1)\}$$
$$= r_1r_2 \{\cos(\varphi_1 + \varphi_2) + j \sin (\varphi_1 + \varphi_2)\} \tag{26}$$

This equation compared with (9) shows that the product of z_1 and z_2 has the product of their magnitudes for a magnitude, and the sum of their angles for an angle.

47. Division of Complex Quantities.

Rule.—Divide the magnitudes and subtract the angles.

Proof.—Using the exponential forms given just above equation (24), we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\epsilon^{j(\varphi_1 - \varphi_2)}.$$

Example.—Divide $r\epsilon^{j\varphi}$ by $a + bj$.

$$a + bj = \sqrt{a^2 + b^2}\epsilon^{j \tan^{-1} (b/a)};$$

whence

$$\frac{r\epsilon^{j\varphi}}{a + bj} = \frac{r}{\sqrt{a^2 + b^2}} \epsilon^{j\{\varphi - \tan^{-1} (b/a)\}} \tag{27}$$

48. To Raise a Complex Quantity to the nth Power.

Rule.—Raise the magnitude to the nth power and take n times the angle.

Proof.—

$$z^n = \{r \epsilon^{j\varphi}\}^n = r^n \epsilon^{jn\varphi}. \tag{28}$$

Example.—

$$(a + bj)^n = (\sqrt{a^2 + b^2})^n \epsilon^{jn \tan^{-1} (b/a)}. \tag{29}$$

49. To Extract the nth Root of a Complex Quantity.

Rule.—Take the nth root of the magnitude and $\frac{1}{n}$*th of the angle.*

Proof.—

$$\sqrt[n]{z} = \sqrt[n]{r\epsilon^{j\varphi}} = \sqrt[n]{r}\epsilon^{j\varphi/n}. \tag{30}$$

50. Integration by the Use of Exponentials.—As an example of the use of the above principles let it be required to integrate $\epsilon^{at} \cos (\omega t + \varphi)$ with respect to t.

Let the abbreviation r.p. be an abbreviation for the words *real part of.*

By equation (13)

$$\cos (\omega t + \varphi) = \text{r.p. } \epsilon^{j(\omega t + \varphi)},$$

whence

$$\int \epsilon^{at} \cos(\omega t + \varphi)dt = \text{r.p.}\int \epsilon^{at + j(\omega t + \varphi)}dt \tag{31}$$

$$= \text{r.p. } \frac{1}{a + j\omega} \epsilon^{(a + j\omega)t + j\varphi},$$

by direct integration,

$$= \text{r.p.}\frac{1}{\sqrt{a^2 + \omega^2}}\epsilon^{(a + j\omega)t + j\varphi - j \tan^{-1} \frac{\omega}{a}},$$

by (27),

$$= \frac{\epsilon^{at}\cos\left(\omega t + \varphi - \tan^{-1}\frac{\omega}{a}\right)}{\sqrt{a^2 + \omega^2}}$$

51. Caution Regarding Use of Antitangents.—In the use of antitangents of ratios such as occur in the preceding problem, it is

important carefully to attend to the signs of quantities occurring in the ratios, for

$$\tan^{-1}(-\omega/a) = -\tan^{-1}(\omega/a), \tag{32}$$

$$\tan^{-1}(\omega/-a) = \pi - \tan^{-1}(\omega/a), \tag{33}$$

$$\tan^{-1}(-\omega/-a) = \pi + \tan^{-1}(\omega/a) \tag{34}$$

52. Problems.—In the following problems $j = \sqrt{-1}$; a, b, r and φ are real quantities.

Abbreviations: r.p. = "real part," i.p. = "imaginary part."

Find r.p and i p of

1. $r\epsilon^{-j\varphi}$.

2. $\dfrac{a_1 + b_1 j}{a_2 + b_2 j}\epsilon^{j\varphi}$.

3. $\sqrt{\dfrac{a_1 + b_1 j}{a_2 + b_2 j}}$; express result in terms of antitangents.

4. Integrate

   $\int \epsilon^{-at} \sin(\omega t + \varphi) dt.$

Prove

5. $\epsilon^{j\pi/2} = j$,

6. $\epsilon^{-j\pi/2} = -j$,

7. $\epsilon^{j\pi} = -1$,

8. $\epsilon^{2j\pi} = 1$.

9. Using exponentials prove that

   $\int \sin x\, dx = -\cos x.$

10. Show that

    $\int \cos^2 \omega t dt$ is not equal to r.p. $\int \left\{\epsilon^{jwt}\right\}^2 dt$.

CHAPTER V

CIRCUIT CONTAINING RESISTANCE, SELF INDUCTANCE, CAPACITY, AND A SINUSOIDAL IMPRESSED ELECTROMOTIVE FORCE

53. Sketch of Method.—If a circuit, Fig. 1, contains in series a resistance R, self inductance L and a capacity C, and has impressed upon it a sinusoidal e.m.f., E sin ωt, the differential equation for the current in the circuit is

$$E \sin \omega t = L\frac{di}{dt} + Ri + \frac{\int idt}{C}. \tag{1}$$

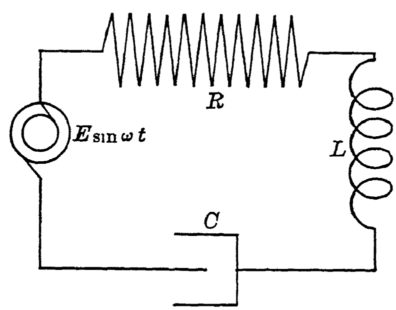


FIG. 1.—Circuit containing sinusoidal e.m.f.

For reference let us write down the equation

$$0 = L\frac{di}{dt} + Ri + \frac{\int idt}{C}. \tag{2}$$

The complete solution of (1) is

(*a*) the complete solution of (2), plus

(*b*) any particular solution of (1).

The proof of this is as follows: Equations (1) and (2) when freed of the integral sign by differentiation are differential equations of the second order. Their general solutions must contain two arbitrary constants, and any solution found for the equation (1) and found to have two arbitrary constants is complete. Now (*a*) reduces the right-hand side of (1) to zero and contains two

arbitrary constants, (*b*) reduces the right-hand side of (1) to its left-hand side; therefore, since the right-hand side of (1) and (2) is homogeneous and of the first degree (linear) the sum of (*a*) and (*b*) reduces the right-hand side of (1) to the left-hand side and contains two arbitrary constants. This sum is, therefore, the complete integral of (1).

We have already found (*b*) the complete integral of (2) in Chapter II, equation (17); namely,

$$i_2 = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t}, \tag{3}$$

where k_1 and k_2 have the values given in equations (i) and (ii) at the beginning of Chapter II, and A_1 and A_2 are two arbitrary constants.

54. The Particular Solution of (1).—It remains only to find a particular solution of (1). To find this let us replace

$$E \sin \omega t \text{ by } E\epsilon^{j\omega t},$$

solve, and take $1/j$ times the imaginary part of the result.

This substitute equation is

$$E\epsilon^{j\omega t} = L\frac{di}{dt} + Ri + \frac{\int idt}{C}. \tag{4}$$

Of this equation we need only a particular solution. This is seen to be of the form

$$i'_1 = A\epsilon^{j\omega t}, \tag{5}$$

as may be seen by direct substitution in (4), giving

$$E\epsilon^{j\omega t} = A\epsilon^{j\omega t}\{Lj\omega + R + \frac{1}{Cj\omega}\}, \tag{6}$$

which is the condition under which (5) is a solution of (4).

This condition (6) reduces to

$$A = \frac{E}{R + j(L\omega - \frac{1}{C\omega})}. \tag{7}$$

Substitution of (7) in (5) gives for the required particular solution of (4)

$$i'_1 = \frac{E\epsilon^{j\omega t}}{R + j(L\omega - \frac{1}{C\omega})}. \tag{8}$$

Adopting the usual notation, let us write an abbreviation

$$X = L\omega - \frac{1}{C\omega}. \tag{9}$$

This quantity X is called the *Reactance of the Circuit.*

In terms of X the denominator of (8) becomes

$$R + j\left(L\omega - \frac{1}{C\omega}\right) = R + jX \tag{10}$$

$$= \sqrt{R^2 + X^2}\epsilon^{j\tan^{-1}\frac{X}{R}}.$$

The last step is taken by the methods of Chapter IV.

Substituting (10) in (8), we have, as the particular solution of (4)

$$i'_1 = \frac{E\epsilon^{j\left(\omega t - \tan^{-1}\frac{X}{R}\right)}}{\sqrt{R^2 + X^2}}. \tag{11}$$

To obtain from this the corresponding particular solution of (1), we need only take the imaginary part of i'_1, and divide it by j, obtaining

$$i_1 = \frac{E}{\sqrt{R^2 + X^2}} \sin \left\{\omega t - \tan^{-1}\frac{X}{R}\right\}. \tag{12}$$

Equation (12) *gives a value of the current* i_1 *that is a particular solution of the differential equation* (1).

55. The General Solution of (1).—We may now obtain the general solution, or complete integral, of (1) by taking the sum of (12) and (3). If we indicate the current by i, we have

$$i = \frac{E}{\sqrt{R^2 + X^2}} \sin \left(\omega t - \tan^{-1}\frac{X}{R}\right) + A_1\epsilon^{k_1 t} + A_2\epsilon^{k_2 t}. \tag{13}$$

Equation (13) *is the complete solution of* (1). *The apparent exception that arises in the critical case in which* $R^2 = 4L/C$ *disappears as an exception after the determination of the arbitrary constants.*

In (13) A_1 *and* A_2 *are arbitrary constants, and*

$$X = L\omega - \frac{1}{C\omega}, \tag{14}$$

$$k_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}, \tag{14a}$$

$$k_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{14b}$$

56. Transformation of the General Solution into Periodic Form.—For some purposes it is more instructive to put the two exponential terms of (13) into the form of a sine function.

This can be done by letting

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \tag{15}$$

$$a_0 = R/2L, \tag{16}$$

then

$$\begin{aligned} k_1 &= -a_0 + j\omega_0, \\ k_2 &= -a_0 - j\omega_0. \end{aligned} \tag{17}$$

With this notation equation (3) becomes

$$\begin{aligned} i_2 &= \epsilon^{-a_0 t}\{A_1(\cos\omega_0 t + j\sin\omega_0 t) + A_2(\cos\omega_0 t - j\sin\omega_0 t)\} \\ &= \epsilon^{-a_0 t}\{(A_1 + A_2)\cos\omega_0 t + j(A_1 - A_2)\sin\omega_0 t\}. \end{aligned} \tag{18}$$

If now in (18) we let

$$\cos\psi_0 = \frac{j(A_1 - A_2)}{\sqrt{(A_1 + A_2)^2 + \{j(A_1 - A_2)\}^2}},$$

$$\sin\psi_0 = \frac{A_1 + A_2}{\sqrt{(A_1 + A_2)^2 + \{j(A_1 - A_2)\}^2}},$$

and

$$I_0 = \sqrt{(A_1 + A_2)^2 + \{j(A_1 - A_2)\}^2},$$

we obtain

$$i_2 = I_0\,\epsilon^{-a_0 t}\sin(\omega_0 t + \psi_0). \tag{19}$$

In equation (19) I_0 and ψ_0 are new arbitrary constants which are to be determined by the initial conditions of the problem. Equation (19) is a perfect equivalent of (3), and after the determination of the arbitrary constants gives correct results whether R^2 is equal to, less than, or greater than $4L/C$; that is, whether ω_0 is zero, real, or imaginary. Only, however, when the angular velocity ω_0 of free oscillation of the circuit is real does the solution remain periodic. If ω_0 is zero or imaginary (19) goes over into the exponential, or hyperbolic, form, which is non-periodic

If we add equation (19) to (12) we obtain the complete expression in the transformed aspect; to wit,

$$\begin{aligned} i = {} & \frac{E}{\sqrt{R^2 + X^2}}\sin\left(\omega t - \tan^{-1}\frac{X}{R}\right) \\ & + I_0\epsilon^{-a_0 t}\sin(\omega_0 t + \psi_0). \end{aligned} \tag{20}$$

Equation (20) *is the complete expression for the current in the circuit containing resistance, self-inductance and capacity, and an impressed sinusoidal e.m.f. This equation is alternative to* (13). *The impressed e.m.f. has angular velocity ω, while ω_0 is*

the angular velocity of free oscillation of the circuit, and $a_0 = R/2L$. I_0 *and* ψ_0 *are arbitrary constants to be determined by the initial conditions.*

57. The Quantity Constituting the Charge of the Condenser.—In equation (20) is given an expression for the current flowing in the circuit under the action of an impressed sinusoidal e.m.f. To obtain q the quantity of electricity constituting the charge of the condenser at any time t, it is only necessary to form the integral

$$\begin{aligned} q &= \int i\,dt \\ &= \frac{-E/\omega}{\sqrt{R^2+X^2}} \cos\left(\omega t - \tan^{-1}\frac{X}{R}\right) \\ &\quad - I_0\sqrt{LC}\,\epsilon^{-a_0 t} \sin\left(\omega_0 t + \psi_0 + \tan^{-1}\frac{\omega_0}{a_0}\right). \end{aligned} \tag{21}$$

58. Determination of the Arbitrary Constants when the E. M.F. is Impressed on a Circuit without Current or Charge.—The reader who is not immediately interested in the determination of these constants may omit this and the next section and resume the reading at the section on the Steady-state Solution (Art. **60**).

In equations (20) and (21) two arbitrary constants I_0 and ψ_0 occur. These are to be determined for each specific problem by the use of the initial conditions.

We cannot in general impose the condition that $t = 0$ when the initial current and charge are zero, for this implies that the dynamo impressing the e.m.f. ($E \sin \omega t$) is thrown into the circuit containing no current and no charge when the dynamo e.m.f. is itself just zero. Now if the dynamo is thrown into the circuit at a random time this will not be the case. Our problem, in case the initial charge and current are zero, imposes the conditions

$$t = t_1,\ i = 0,\ q = 0, \tag{22}$$

where t_1 is the random time determining the phase of the e.m.f. at the time of impressing it.

If now, for abbreviations, we let

$$\left.\begin{aligned} I &= \frac{E}{\sqrt{R^2+X^2}} \\ \varphi &= \omega t - \tan^{-1}\frac{X}{R}, \\ \text{and} \qquad \varphi_1 &= \omega t_1 - \tan^{-1}\frac{X}{R}, \end{aligned}\right\} \tag{23}$$

equations (20) and (21) become

$$i = I \sin \varphi + I_0 \epsilon^{-a_0 t} \sin(\omega_0 t + \psi_0), \tag{24}$$

and

$$q = -\frac{I}{\omega} \cos \varphi - I_0 \sqrt{LC}\, \epsilon^{-a_0 t} \sin\left(\omega_0 t + \psi_0 + \tan^{-1} \frac{\omega_0}{a_0}\right), \tag{25}$$

where it is to be noticed that φ is a function of the time t.

The initial conditions (22) introduced into (24) and (25) give

$$0 = I \sin \varphi_1 + I_0 \epsilon^{-a_0 t_1} \sin(\omega_0 t_1 + \psi_0) \tag{26}$$

and

$$0 = -\frac{I}{\omega} \cos \varphi_1 - I_0 \sqrt{LC}\, \epsilon^{-a_0 t_1} \sin\left(\omega_0 t_1 + \psi_0 + \tan^{-1} \frac{\omega_0}{a_0}\right). \tag{27}$$

[illegible] I_0 between these two equations by transposing the first term of each equation to the left-hand side and dividing (27) by (26), we obtain

$$\frac{1}{\omega} \cot \varphi_1 = \sqrt{LC} \frac{\sin\left(\omega_0 t_1 + \psi_0 + \tan^{-1} \frac{\omega_0}{a_0}\right)}{\sin(\omega_0 t_1 + \psi_0)}$$

$$= LC \frac{a_0 \sin(\omega_0 t_1 + \psi_0) + \omega_0 \cos(\omega_0 t_1 + \psi_0)}{\sin(\omega_0 t_1 + \psi_0)}$$

$$= a_0 LC + \omega_0 LC \cot(\omega_0 t_1 + \psi_0), \tag{28}$$

whence

$$\cot(\omega_0 t_1 + \psi_0) = \frac{1}{\omega_0 \omega LC} \cot \varphi_1 - \frac{a_0}{\omega_0}$$

$$= \frac{a_0^2 + \omega_0^2}{\omega \omega_0} \cot \varphi_1 - \frac{a_0}{\omega_0}.$$

We may now use the general trigonometric relation

$\sin x = \dfrac{1}{\sqrt{1 + \cot^2 x}}$, and from the preceding equation obtain

$$\sin(\omega_0 t_1 + \psi_0) = \frac{1}{\sqrt{1 + \left\{\frac{a_0^2 + \omega_0^2}{\omega \omega_0} \cot \varphi_1 - \frac{a_0}{\omega_0}\right\}^2}} = P \text{ (say).} \tag{29}$$

Now the quantity P, defined as equal to the middle term of (29) is completely given in terms of the constants of the the circuits (a_0 and ω_0), the angular velocity of the impressed e.m.f. (ω), and the time at which the e.m.f. is impressed [comprised in φ_1 defined in (23)].

To determine the two constants of integration, we have by (29)

$$\psi_0 = \sin^{-1}P - \omega_0 t_1,$$

and by (29) and (26)

$$I_0 = \frac{-I}{P}\epsilon^{+a_0 t_1}\sin\varphi_1.$$

Substituting these constants into (20) we have, by (23),

$$i = \frac{E}{\sqrt{R^2+X^2}}\left[\sin\left\{\omega t - \tan^{-1}\frac{X}{R}\right\} - \frac{\sin\varphi_1}{P}\epsilon^{-a_0(t-t_1)}\sin\{\omega_0(t-t_1)+\sin^{-1}P\}\right]. \quad (30)$$

Equation (30) *gives the complete value of the current i when the e.m.f. is impressed at a time t_1 upon a circuit without current or charge. In this equation φ_1 and P have the values given in* (23) *and* (29). *In the expression for i, t is greater than t_1, which is the time at which the e.m.f. E sin ωt was thrown into the circuit.*

59. Condition That Makes the Transient Term in (30) Zero.—The term involving the exponential in (30) is called *the transient term.*

One method of making this transient term zero is to let t be infinite. We shall consider this in the next section.

Another method of making this transient term zero in equation (30), may be discovered by expanding it and setting the expanded value equal to zero, obtaining

$$\frac{\sin\varphi_1}{P}\left\{\sqrt{1-P^2}\sin\omega_0(t-t_1) + P\cos\omega_0(t-t_1)\right\} = 0.$$

Each of these terms must be zero, hence

$$\frac{\sin\varphi_1}{P}\sqrt{1-P^2} = 0 \quad (31)$$

and

$$\sin\varphi_1 = 0. \quad (32)$$

These two conditions must both be fulfilled. The first of these by (29) gives

$$\sin\varphi_1\left\{\frac{a_0^2+\omega_0^2}{\omega\omega_0}\cot\varphi_1 - \frac{a_0}{\omega_0}\right\} = 0.$$

That is

$$\frac{a_0^2+\omega_0^2}{\omega\omega_0}\cos\varphi_1 - \frac{a_0}{\omega_0}\sin\varphi_1 = 0.$$

Since, however, $\sin \varphi_1 = 0$, we have

$$a_0^2 + \omega_0^2 = 0 = 1/LC \text{ by (15).}$$

Hence

$$L = \infty \text{ or } C = \infty \tag{33}$$

Replacing φ_1 in (32) by its value from (15) we have

$$\sin \left\{ \omega t_1 - \tan^{-1} \frac{X}{R} \right\} = 0. \tag{34}$$

Under these conditions (33) and (34) the complete current is

$$i = \frac{E}{\sqrt{R^2 + X^2}} \sin \left(\omega t - \tan^{-1} \frac{X}{R} \right). \tag{35}$$

Equation (33) *shows that transients can be avoided only provided* $L = \infty$ *or* $C = \infty$. *Under either of these conditions Equation* (34) *gives the time* t_1 *at which the sinusoidal e.m.f. may be impressed without any transient in the resulting current, and* (35) *is the resulting current.*

The case of $C = \infty$ *is of the greater practical importance,* for it is *the case of a short-circuited condenser, and hence the case of a resistive inductance thrown into series with an applied e.m f.*

60. Results in the Steady State.—Apart from the method outlined in the preceding section for making the transient term in the current equation zero, it is seen that this transient term in each case is multiplied by an exponential factor with an exponent that approaches minus infinity with increase of time. If the time is sufficiently long after the application of the sinusoidal e.m.f., the transient term becomes negligible.

The state of things after the transient term has become practically zero is called *the steady state,* and the solution for the steady state is called *the steady state solution.*

In the steady state, after the transient term has become practically zero, it is seen from (20) and (21) that the current and quantity are given by the equations

$$i = \frac{E}{\sqrt{R^2 + X^2}} \sin \left(\omega t - \tan^{-1} \frac{X}{R} \right), \tag{36}$$

$$q = \frac{-E/\omega}{\sqrt{R^2 + X^2}} \cos \left(\omega t - \tan^{-1} \frac{X}{R} \right), \tag{37}$$

in which

$$E \sin \omega t = \text{the impressed e.m.f., and}$$
$$X = L\omega - 1/C\omega = \text{the reactance of the circuit.} \qquad (38)$$

R, L, and C = the resistance, inductance, and capacity of the circuit.

Equations (36) *and* (37) *give the values of the current i and the quantity of electricity q constituting the charge of the condenser at the time t, under the action of a sinusoidal e.m.f.E sin ωt which has been in application sufficiently long to permit the establishment of a steady state.*

Some of the characteristics of the steady-state flow of current will be discussed in the next Chapter on Electrical Resonance in Simple Circuits.

CHAPTER VI

ELECTRICAL RESONANCE IN A SIMPLE CIRCUIT

61. Wave Length, Actual and Conventional.—We have seen in Chapter II that an electrical circuit containing capacity and self-inductance, if the resistance is not too great, has a characteristic period of oscillation. We shall show in subsequent chapters, treating Maxwell's Electromagnetic Theory that, with certain forms of these circuits, energy is radiated into surrounding space in the form of electromagnetic waves.

If a circuit of period T radiates waves, the wave length λ of the waves radiated is related to the period T by the equation

$$\lambda = cT, \tag{1}$$

where

$$\lambda = \text{wave length},$$

and

$$c = \text{velocity of propagation of the waves.}$$

This relation follows from the elementary consideration that of two successive positive wave crests one is emitted at a time T seconds later than the other. The first, in the time T, travels a distance cT, so that the first crest is a distance cT ahead of the second; hence the distance between these two successive positive wave crests, which is the wave length, is $\lambda = cT$.

In free space, we shall show from Maxwell's Theory, that c, the velocity of the waves in free space is the velocity of light; that is, $c = 3 \times 10^{10}$ centimeters per second. If it is required to obtain the wave length in meters, as is usual in radiotelegraphic practice, and if T is in seconds, the velocity of propagation must be expressed in meters per second; that is

$$c = 3 \times 10^{8} \text{ meters per second.} \tag{2}$$

In the case of an actual radiation of electric waves into space, the wave length λ is the actual distance between adjacent positions of similar phase in the emitted wave system.

It has become customary in radiotelegraphic practice to specify

the period of all periodic electric circuits in terms of the wave lengths corresponding to the periods of the circuits, even when the circuits happen to be of such form as actually to radiate only an insignificant amount of energy as characteristic waves. We thus attribute conventionally to every oscillatory circuit a wave length λ satisfying the relation (1).

Although we have not yet taken up the matter of electromagnetic radiation, it is often an advantage to express results in terms of wave lengths as well as in terms of periods, and to use, in experimental investigations with these circuits, apparatus calibrated in wave lengths.

62. Mean Square Current and Amplitude of Current in a Circuit Containing Resistance, Self-inductance, and Capacity, and a Sinusoidal E.M.F.—The circuit upon which the e.m.f. is

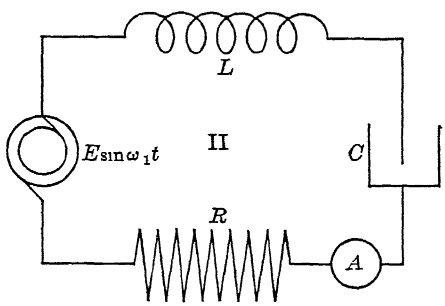


FIG. 1.—Circuit containing impressed sinusoidal e.m.f.

impressed we shall designate as Circuit II, or as *the Receiving Circuit.* The e.m.f. may be impressed by a generator in the circuit (see Fig. 1), or it may be impressed by induction from a Circuit I (Fig. 2), containing persistent sinusoidal oscillations, provided the Circuit I be so far from the Circuit II that the reaction of Circuit II in changing the current in Circuit I is negligible. The subject of these reactions will be taken up in Chapters VII and VIII, but the reactions will here be considered zero.

Let the e.m.f. impressed on II be

$$e = E \sin \omega_1 t. \tag{3}$$

Let the resistance, inductance and capacity of the receiving circuit (Circuit II) be R, L, and C, and, as in the previous chapters, let the capacity be disposed in one or more discrete condensers so that there is no distributed capacity.

Then, after a steady state is reached, the current in II, designated by i, is, by (36), Chapter V,

$$i = \frac{E}{\sqrt{R^2 + X^2}} \sin\left(\omega_1 t - \tan^{-1}\frac{X}{R}\right), \tag{4}$$

where

$$X = L\omega_1 - 1/C\omega_1. \tag{5}$$

Since many types of measuring instruments, when placed at A in series in Circuit II, indicate the average square of the current or else the square root of the mean square current (R.M.S. current), let us obtain the value of these quantities. First let

$$y = \sin(\omega t + \varphi)$$

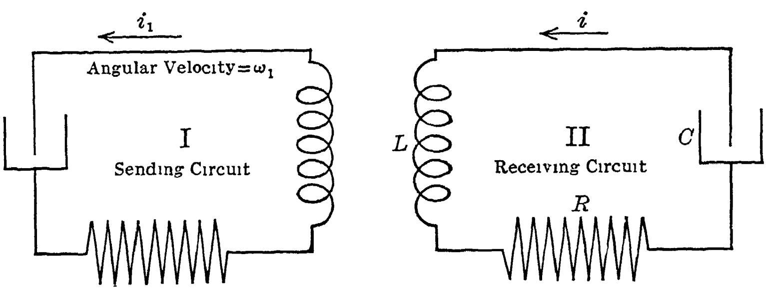


FIG. 2—Arrangement for inducing sinusoidal e m f in II, if I is far enough away and is traversed by a sinusoidal current

and note that the time average of a quantity during the interval from 0 to t' is obtained by integrating the quantity with respect to t from 0 to t' and dividing the integral by t'.

According to this principle the time average of y^2, which we shall indicate by

$$\overline{y^2} = \text{mean square value of } y,$$

is

$$\begin{aligned}\overline{y^2} &= \frac{1}{t'}\int_0^{t'} \sin^2(\omega t + \varphi)dt \\ &= \frac{1}{t'}\int_0^{t'} \left\{\frac{1 - \cos 2(\omega t + \varphi)}{2}\right\} dt \\ &= \frac{1}{t'}\left[\frac{t}{2}\right]_0^{t'} - \left[\frac{\sin 2(\omega t + \varphi)}{4\omega t'}\right]_0^{t'}.\end{aligned}$$

If t' is very large compared with the half period of oscillation, or

if t' is exactly a whole number of half cycles of y, the second term on the right disappears, and

$$\overline{y^2} = 1/2 \tag{7}$$

Equation (7) gives the mean square value of y, taken over n half cycles or over any time long in comparison with the period of y, where

$$y = \sin (\omega t + \varphi). \tag{8}$$

Returning now to an investigation of (4) let us note that the time average of the sine term is 1/2, so that

$$\overline{I^2} = \frac{E^2/2}{R^2 + X^2}, \tag{9}$$

where $\overline{I^2}$ is the mean square (time average of the square) of the current in II.

If now we look back at the value of the impressed e.m.f. (3), we see that the time average of the square of the e.m.f. to be designated by $\overline{E^2}$ is

$$\overline{E^2} = E^2/2, \tag{10}$$

which may replace the numerator in (9) giving

$$\overline{I^2} = \frac{\overline{E^2}}{R^2 + X^2}. \tag{10}$$

Instead of using the mean square value of E and I, as in equation (10), we may as an alternative operation express the *amplitude* of I in terms of the *amplitude* of E, in the same form of equation; namely, by (4), making the sine term unity,

$$I^2 = \frac{E^2}{R^2 + X^2}. \tag{11}$$

Equation (10) gives the mean square value of current, $\overline{I^2}$, in terms of the mean square value of impressed e.m.f., $\overline{E^2}$, in the steady state. Equation (11) gives the corresponding equation for the amplitude I of current in terms of the amplitude E of e.m.f.

63. Condition for Steady-state Current-resonance in a Simple Circuit Containing a Sinusoidal Impressed E.M.F.—The steady-state current resonance condition is defined as the relation between the constants of the circuit and the frequency of the impressed e.m.f. for which the mean square current or current amplitude is a maximum, when the amplitude of the e.m.f. is constant.

By (10) and (11) it is seen that this condition is

$$X = 0; \tag{12}$$

that is, by (5)

$$LC = 1/\omega_1^2. \tag{13}$$

By taking 2π times the square root of both sides of (13) we obtain for the current-resonance condition

$$2\pi\sqrt{LC} = T_1, \tag{14}$$

where

$$T_1 = 2\pi/\omega_1 = \text{period of impressed e.m.f}$$

Note that in (14) while the right-hand side is the period of the impressed e.m.f., the left-hand side, by (58), Chapter II, is the *undamped period of the receiving circuit* (Circuit II) so we may conclude that

The condition for a maximum mean square current or the condition for a maximum amplitude of current in the steady state, which condition we have called the Current-resonance Condition, is that the Undamped Period of the Receiving Circuit (not the actual free period) be equal to the actual period T_1 of the impressed e.m.f.

64. Steady-state Value of Current at Current-resonance.— At current-resonance in the steady state the current is obtained by setting $X = 0$ in (10) or (11), and extracting the square root. This gives

$$I_{\text{max}} = \frac{E}{R} \tag{15}$$

where I and E are either amplitude values or square root of mean-square values (R.M.S. values).

By reference to (4) it will be seen that the angle by which the current lags behind the impressed e.m.f. at current-resonance is zero, since $X = 0$.

Hence, also, at current-resonance, by (3), (4), and (12), we have

$$i = e/R \tag{16}$$

where i and e are instantaneous values.

In the steady state at current-resonance the relation of current to impressed e.m.f. is Ohm's Law.

In this condition the inductive reactance $L\omega_1$ and the capacity reactance $-1/C\omega_1$ are numerically equal to each other and opposite in sign and are sometimes said to neutralize each other.

65. Ratio of Current in the General Case to Current at Current-resonance.—Let us now divide (11) by the square of (15) obtaining

$$\frac{I^2}{I^2_{\text{max.}}} = y(\text{say}) = \frac{1}{1 + X^2/R^2} \tag{17}$$

This equation is equally true whether I^2 and $I^2_{\text{max.}}$ are the squares of the amplitudes of current or the mean-square values, since the ratio of amplitudes squared and the ratio of the time average of the squares of instantaneous values are the same.

If, in (17), we replace X by its value from (5), we obtain

$$y = \frac{1}{1 + \frac{L^2\omega_1^2}{R^2}\left\{1 - \frac{1}{LC\omega_1^2}\right\}^2} \tag{18}$$

where

$$y = I^2/I^2_{\text{max.}} = \overline{I^2}/\overline{I^2}_{\text{max}} \tag{19}$$

Equation (18) *is the equation to a resonance curve of current square against the circuit adjustments.*

We can apply (18) *to specific cases in which different elements of the system are variable. We shall discuss two such cases.*

66. Resonance Curve of Relative Current Square with a Fixed Impressed E.M.F. and Variation of Capacity in the Receiving Circuit.—Referring to Fig. 1 or Fig. 2, we have called the circuit II, with constants L, R, and C, the *receiving circuit.* Impressed upon Circuit II is a sinusoidal e.m.f. of value

$$e = E \sin \omega_1 t,$$

in which ω_1 is the angular velocity of impressed e.m.f. We shall now suppose that ω_1 and E are kept constant, and we shall compute the relative current square in the receiving circuit when the condenser C of the receiving circuit is given various values.

The fundamental equation of the result is given in (18), and we shall merely transform this equation into a form involving wavelengths and decrements instead of inductances, capacities, resistances, and angular velocities.

Regarding the decrement, we have defined in (62) of Chapter II a quantity

$$d = \frac{RT}{2L} \tag{20}$$

whose period of free oscillation is T, and whose resistance and inductance are R and L.

Now the period of free oscillation of a circuit is exactly given in (56), of Chapter II. This period is given approximately in (58), Chapter II; namely

$$T = 2\pi\sqrt{LC} \tag{21}$$

Although (21) is only an approximate value of the free period of oscillation of the circuit, it is the exact value of the *Undamped Period of the Circuit.*

We shall, accordingly define a new logarithmic decrement, indicated by δ, with the exact equation

$$\delta = \frac{R2\pi\sqrt{LC}}{2L} \tag{22}$$

and shall designate this decrement δ *as the logarithmic decrement per undamped period of the circuit.*

Since we are going to vary C in the present article, δ as defined in (22) is a variable. Let us fix our attention on one particular value of δ, namely the value of δ when C has the value to give a maximum value of y, and designate this value of δ as δ_0. Now by (18) for a maximum of y, it is seen that

$$LC_0 = 1/\omega_1^2 \tag{23}$$

where

C_0 = value of C that makes y a maximum.

From (22) and (23), we have

$$\delta_0 = \frac{R2\pi}{2L\omega_1} = \frac{\pi R}{L\omega_1}, \tag{24}$$

where

δ_0 = logarithmic decrement per undamped period at current-resonance.

Let us next examine the question of wavelengths. The period of the impressed e.m.f. T_1 (say) is related to ω_1 by the equation

$$T_1 = 2\pi/\omega_1 \tag{25}$$

According to equation (1) the wavelength λ_1 of the impressed e.m.f. is

$$\lambda_1 = cT_1 = 2\pi c/\omega_1 \tag{26}$$

If we call the period of the circuit T, the wavelength of the circuit λ is

$$\lambda = cT \tag{27}$$

where

c = velocity of propagation of the waves.
T = free period of oscillation of the circuit.

Since T is not exactly given by (21), while the undamped period of the circuit is exactly given by (21), let us define the *undamped wavelength* of the circuit as the wavelength of the

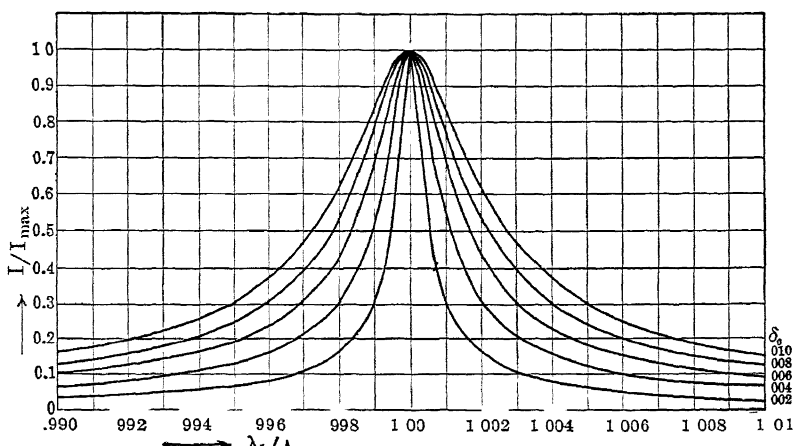


FIG. 3.—Curves of relative current vs relative wavelengths for various values of decrement

undamped period, and indicate this undamped wavelength by a Greek capital Lambda Λ, then

$$\Lambda = c2\pi\sqrt{LC} \tag{28}$$

In general when the circuits have small decrements Λ does not differ appreciably from λ, but when the decrements are large, we should find it inaccurate to replace Λ by λ.

If now we substitute (24), (26), and (28) into (18), we obtain

$$y = \frac{1}{1 + \frac{\pi^2}{\delta_0{}^2}\left\{1 - \left(\frac{\lambda_1}{\Lambda}\right)^2\right\}^2} \tag{29}$$

Equation (29) *gives the value of relative square-current* y, *as defined in* (19), *in terms of the undamped wavelength* Λ *of the re-*

ceiving circuit, for a fixed value of the wavelength λ_1 of the impressed e m f.

67. Sample Curves of Relative Current for Fixed Impressed E.M.F. and Variation of the Capacity of the Receiving Circuit.— If we extract the square root of (29) we have

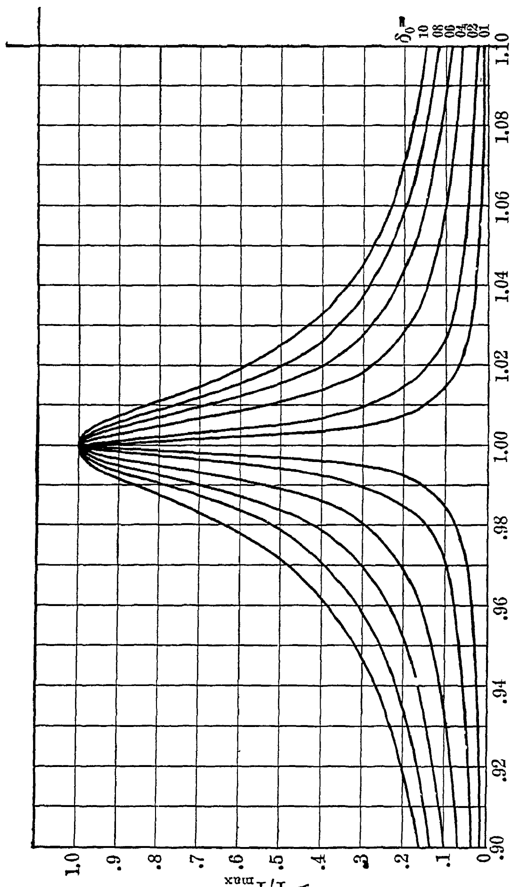


FIG. 4.—Same as Fig. 3, but with different values of δ_0 and different horizontal scale.

$$\frac{I}{I_{\max}} = \frac{1}{\sqrt{1 + \frac{\pi^2}{\delta_0{}^2}\left\{1 - \frac{\lambda_1{}^2}{\Lambda^2}\right\}^2}} \tag{30}$$

Equation (30) *is true whether I and $I_{max.}$ are amplitude values or R.M.S. Values.*

Figures 3, 4, 5, and 6 contain plots of equation (30) for different values of δ_0. These curves were traced from blue prints

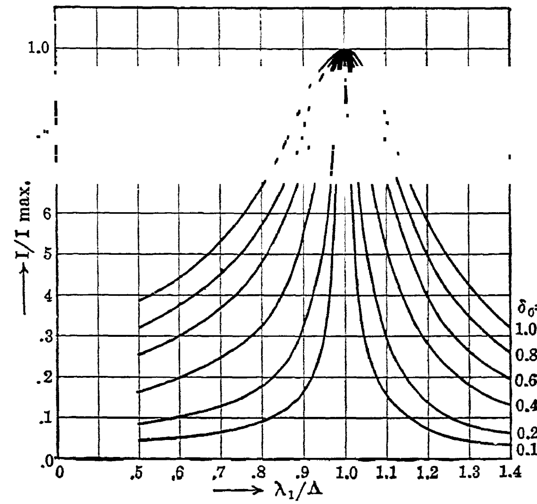


FIG. 5.—Same as Fig 3, but with different values of δ_0 and different horizontal scale.

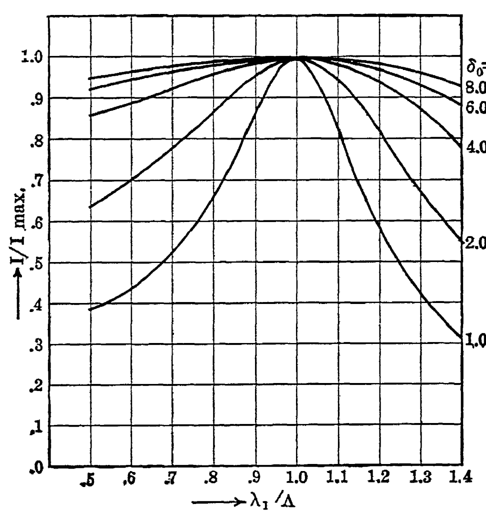


FIG. 6.—Same as Fig. 5, but with different values of δ_0.

kindly supplied to me by Mr. J. Martin, Expert Radio Aide of the U. S. Navy.

68. Determination of Decrement from Relative Current Square with Fixed E.M.F. and Variation of C.—From (29) we may obtain

$$\delta_0 = \frac{\pm \pi \{1 - \lambda_1^2/\Lambda^2\}}{\sqrt{\frac{1}{y} - 1}}, \tag{31}$$

in which

$$y = I^2/I^2_{\text{max}} = \overline{I^2}/\overline{I^2}_{\text{max.}}$$
$$\lambda_1 = \text{wavelength of impressed e.m.f., and}$$
$$\Lambda = \text{undamped wavelength of the circuit.}$$

By plotting a curve of y vs. λ_1^2/Λ^2, we may compute a value of δ_0 for every value of Λ. All of the values of δ_0 so obtained should agree within the limits of accuracy of the measurements. It is apparent that this accuracy is not very great, but fortunately it is not generally of importance to know δ_0 with great accuracy.

69. Approximate Method of Rapidly Determining δ_0.—As an approximate method of determining δ_0, let Λ' and Λ'' = the two values of Λ at which y has the value $\frac{1}{2}$, and let $\Lambda'' > \Lambda'$, then by (31)

$$\delta_0 = \pi(1-\lambda_1^2/\Lambda''^2) \text{ and}$$
$$\delta_0 = -\pi(1-\lambda_1^2/\Lambda'^2).$$

Adding these equations and dividing by 2, we may take the following steps

$$\delta_0 = \frac{\pi\lambda_1^2}{2}\left\{\frac{1}{\Lambda'^2} - \frac{1}{\Lambda''^2}\right\}$$
$$= \frac{\pi\lambda_1^2}{2\Lambda'^2\Lambda''^2}\{\Lambda''^2 - \Lambda'^2\}$$
$$= \frac{\pi\lambda_1^3\,(\Lambda'' + \Lambda')}{2\Lambda'^2\,\Lambda''^2}\left\{\frac{\Lambda'' - \Lambda'}{\lambda_1}\right\}.$$

We may now introduce approximations as follows:
Let

$$\Lambda'\Lambda'' = \lambda_1^2, \tag{32}$$

and

$$\Lambda' + \Lambda'' = 2\lambda_1 \text{ approximately}, \tag{33}$$

then

$$\delta_0 = \pi\left\{\frac{\Lambda'' - \Lambda'}{\lambda_1}\right\}, \text{ approximately.} \tag{34}$$

To the same degree of approximation $\delta_0 = d$.

We may state this result in the following rule.

70. Rule for Approximate Determination of Logarithmic Decrement d of a Circuit with Variable Capacity.—*To obtain the logarithmic decrement of a circuit, impress upon it an undamped e.m f. of constant amplitude and frequency, take the difference of the two wavelength adjustments of the circuit that give a mean square current equal to half the maximum mean square current, divide this difference by the wavelength adjustment of the circuit that gives a maximum mean square current, and multiply the quotient by π. This gives δ_0 which is approximately d.*

71. Problem.—For practice it is recommended that the reader apply this rule to the curves of Figs. 3, 4, 5, and 6, noting that the ordinates of these curves are the square roots of y, and that for y to fall to a half value, the square root of y falls to .707 times the maximum value.[1]

78. Determination of Decrement by Impressing an Undamped E.M.F. of Fixed Amplitude and Variable Frequency on a Circuit of Fixed Inductance, Capacity and Resistance.—The starting point for this paragraph is the general equation (18).

The decrement per undamped period of the fixed circuit is given in (22), from which we obtain

$$\frac{L^2}{R^2} = \frac{\pi^2 LC}{\delta^2}. \tag{35}$$

This substituted into (18) gives

$$y = \frac{1}{1 + \frac{\pi^2}{\delta^2} LC\omega_1^2 \left\{ 1 - \frac{1}{LC\omega_1^2} \right\}^2}. \tag{36}$$

Now introducing the wavelength values given in (26) and (28) we obtain

$$y = \frac{1}{1 + \frac{\pi^2}{\delta^2} \frac{\Lambda^2}{\lambda_1^2} \left\{ 1 - \frac{\lambda_1^2}{\Lambda^2} \right\}^2}. \tag{37}$$

$$= \frac{1}{1 + \frac{\pi^2}{\delta^2} \frac{\lambda_1^2}{\Lambda^2} \left\{ \frac{\Lambda^2}{\lambda_1^2} - 1 \right\}^2}$$

This last equation, solved for δ gives

$$\delta = \frac{\pm \pi\lambda_1}{\Lambda} \frac{\Lambda^2/\lambda_1^2 - 1}{\sqrt{1/y - 1}}, \tag{38}$$

in which the sign is to be chosen so as to make δ positive.

[1] Absence of sections numbered **72** to **77** has no significance.

Equation (38) gives the decrement δ per undamped period of the fixed circuit upon which is impressed an e.m.f. of constant amplitude and of wavelength λ_1. The undamped wavelength of the fixed circuit is Λ defined by (28).

79. Approximate Method for Rapidly Determining δ with Fixed Circuit and Variable Impressed Angular Velocity.—Analogously to the case of fixed e m.f. and variable circuit, as approximately treated in Art. **69**, we may treat approximately the case of fixed circuit with a variable frequency of impressed e.m.f.

Let λ'_1 and λ''_1 = impressed wavelengths at which y has half the maximum value,

then by (38), if $\lambda''_1 > \lambda'_1$, we have

$$\delta = \frac{\pi\lambda'_1}{\Lambda}\left(\frac{\Lambda^2}{\lambda'^2_1} - 1\right), \tag{39}$$

and

$$\delta = -\frac{\pi\lambda''_1}{\Lambda}\left(\frac{\Lambda^2}{\lambda''^2_1} - 1\right). \tag{40}$$

Adding these two equations, dividing by 2, and factoring, we obtain

$$\delta = \frac{\pi}{2}(\lambda''_1 - \lambda'_1)\left\{\frac{\Lambda}{\lambda'_1\lambda''_1} + \frac{1}{\Lambda}\right\}. \tag{41}$$

Equation (41) is exact.

Now λ''_1 is greater than Λ and λ'_1 is less than Λ, so that if λ''_1 and λ'_1 are not too far apart, their product is approximately equal to Λ^2, so that (41) reduces to

$$\delta = \pi\left\{\frac{\lambda''_1 - \lambda'_1}{\Lambda}\right\}, \text{ approximately, } = d, \text{ approximately.} \tag{42}$$

This result may be stated in the following rule.

80. Rule for Approximate Determination of Logarithmic Decrement d, with Circuit Fixed and Frequency of Impressed E.M.F. Varied.—*To obtain the logarithmic decrement of a circuit of fixed constants, impress upon it an e.m.f. of fixed amplitude variable as to frequency. Take the difference of the two impressed wavelengths that produce a mean-square current equal to half the maximum mean-square current, divide this difference by the wavelength that gives a maximum mean-square current, and multiply the quotient by π.*

CHAPTER VII

THE FREE OSCILLATION OF TWO COUPLED RESISTANCELESS CIRCUITS. PERIODS AND WAVELENGTHS[1]

81. Differential Equations for Inductively Coupled System of Two Circuits.—If we have two circuits, as in Fig. 1, with the inductances of the two circuits near enough together to permit currents flowing in one of the circuits to induce electromotive forces of appreciable values in the other circuit, the circuits are said to be coupled.

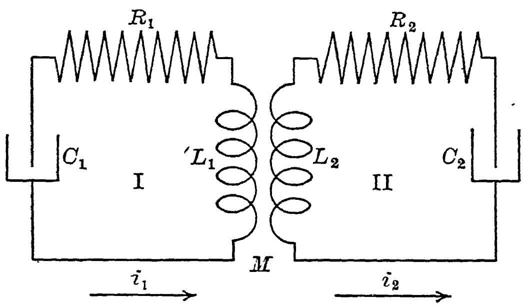


FIG. 1.—Two circuits I and II, coupled by a transformer.

In the illustration the coupling is by mutual induction and is said to be inductive coupling.

In setting up the differential equations both circuits will be assumed to have inductance, capacity, and resistance. The electromotive force impressed upon the system from without is supposed to be zero.

[1] The following is a partial list of references to theoretical works on the free oscillation of two coupled circuits:

Lord Rayleigh, "Theory of Sound;" J. von Geitler, *Sitz. d. k. Akad. d. Wiss. z. Wien*, February and October, 1905; B. Galizine, *Petersb. Ber.*, May and June, 1895; V. Bjerkness, *Wied. Ann.*, 55, p. 120, 1895; Oberbeck, *Wied. Ann.*, 55, p. 625, 1895; Domalip and Kolaček, *Wied. Ann.*, 57, p. 731, 1896; M. Wien, *Wied. Ann.*, 61, p. 151, 1897, and *Ann. d. Phys.*, 8, p. 686, 1902; Drude, *Ann. d. Phys.*, 13, p. 512, 1904; B. Macku, *Jahrb. d. drahtlos. Teleg.*, 2, p. 251, 1909; Cohen, *Bul. Bu. of Standards*, 5, p. 511, 1909.

Independent of the method of setting up the currents in the system, the current i_1 flowing in the Circuit I induces an electromotive force $M\frac{di_1}{dt}$ in Circuit II, and likewise the current i_2 flowing in Circuit II induces an electromotive force $M\frac{di_2}{dt}$ in Circuit I, where

M = mutual inductance between the two circuits.

Consequently the differential equations for the currents in the two circuits are

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{\int i_1 dt}{C_1} = M\frac{di_2}{dt}, \tag{1}$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{\int i_2 dt}{C_2} = M\frac{di_1}{dt}, \tag{2}$$

where

L_1, R_1, C_1 = respectively inductance, resistance and capacity of Circuit I,

L_2, R_2, C_2 = corresponding values for Circuit II

82.[1] **Differential Equations for a Direct Coupled System of Two Circuits.**—In the inductively coupled system described

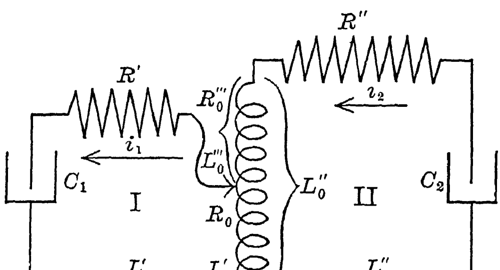


FIG 2.—Two circuits I and II coupled by an auto-transformer

above the two coils L_1 and L_2, which acted mutually upon each other, had no part of their metallic circuits in common. The mutual action between them was by means of the transformer with separate and distinct primary and secondary coils.

Circuits are also often connected by an auto-transformer, as in Fig. 2, where the two circuits have a metallic part L'_0 in common. This connection is called a direct connection or direct

[1] This article is somewhat confusing and may be omitted at first reading.

coupling. It will now be shown that this system leads to a set of differential equations that under certain conditions are the same as the equations for the inductively coupled system.

For the sake of generality we may suppose that certain coils of the system, as L' and L'', have no mutual action upon each other or upon other parts of the system, while other coils, as L'_0 and L''_0 do have mutual induction.

Let M = the mutual inductance between these two coils, L'_0 and L''_0,

M' = the mutual inductance between L'_0 and L'''_0, where L'''_0 is the part of the coil L''_0 which is not common to L'_0.

Let

$$\left.\begin{aligned} L_1 &= L' + L'_0, \\ R_1 &= R' + R_0, \\ L_2 &= L'' + L''_0 = L'' + L'_0 + L'''_0 + 2M', \\ R_2 &= R'' + R_0 + R'''_0. \end{aligned}\right\} \tag{3}$$

Then as before L_1 and L_2 are the total self-inductances of the Circuits I and II respectively, and R_1 and R_2 are the total resistances, and M the total mutual inductance.

Now taking the counter e.m.f.'s around the two circuits, noting that the coil L'_0 is traversed by a current $i_1 - i_2$, we have

$$L'\frac{di_1}{dt} + R'i_1 + \frac{\int i_1 dt}{C_1} + R_0(i_1 - i_2) + L'_0\frac{d}{dt}(i_1 - i_2) - M'\frac{di_2}{dt} = 0, \tag{4}$$

$$L''\frac{di_2}{dt} + R''i_2 + \frac{\int i_2 dt}{C_2} + R'''_0 i_2 + R_0(i_2 - i_1) + L'''_0\frac{di_2}{dt} + L'_0\frac{d}{dt}(i_2 - i_1) + M'\frac{di_2}{dt} + M'\frac{d}{dt}(i_2 - i_1) = 0 \tag{5}$$

Equations (4) *and* (5) *are the differential equations for the currents i_1 and i_2 in the two circuits respectively when the two circuits are connected by having mutual inductance, and part of a coil in common.*

Introducing the values of L_1, L_2, R_1 and R_2 from (3), we obtain from (4) and (5)

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{\int i_1 dt}{C_1} = (M' + L'_0)\frac{di_2}{dt} + R_0 i_2, \tag{6}$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{\int i_2 dt}{C_2} = (M' + L'_0)\frac{di_1}{dt} + R_0 i_1. \tag{7}$$

Now

$$M' + L'_0 = M \tag{8}$$

as may be seen by the following considerations. M is the magnetic flux linkage common to L'_0 and L''_0 for a unit current in L'_0, which is the linkage with itself $(=L'_0)$ plus the linkage with L_0'' $(=M')$.

Substituting (8) in (6) and (7) we have

$$L_1\frac{di_1}{dt} + R_1i_1 + \frac{\int i_1dt}{C_1} = M\frac{di_2}{dt} + R_0i_2, \tag{9}$$

$$L_2\frac{di_2}{dt} + R_2i_2 + \frac{\int i_2dt}{C_2} = M\frac{di_1}{dt} + R_0i_1. \tag{10}$$

Equations (9) *and* (10) *are the differential equations for the currents* i_1 *and* i_2 *in the two circuits respectively, when the two circuits are direct coupled.* R_0 *is the resistance of the element common to the two circuits.*

It is seen that these two equations are identical with those (1) *and* (2) *for the inductively coupled circuits, provided the resistance of that part of the coil common to the two circuits is negligible.*

It is evident that various other methods of coupling[1] the circuits together may be employed; for example, they may be connected together by having a condenser in common, but we shall at present confine our attention to the two types of coupling here illustrated, and shall proceed to treat the special case in which all the resistances of the two circuits are negligible.

We shall describe both types of circuits here illustrated as *magnetically coupled.*

83. Differential Equations for Two Magnetically Coupled Circuits of Negligible Resistances.—If all of the resistances of the two circuits are negligible, the equations (1) and (2) for the inductively coupled circuits and the equations (9) and (10) for the direct coupled circuits reduce to the form

$$L_1\frac{di_1}{dt} + \frac{\int i_1dt}{C_1} = M\frac{di_2}{dt}, \tag{11}$$

$$L_2\frac{di_2}{dt} + \frac{\int i_2dt}{C_2} = M\frac{di_1}{dt}. \tag{12}$$

These are the differential equations in the resistanceless case of two magnetically coupled circuits.

[1] See subsequent chapters.

84. Steps toward a Solution of (11) and (12).—The two equations (11) and (12) are to be solved as simultaneous. The elimination of one of the i's from those two equations will give a homogeneous linear differential equation of the fourth order[1] in the other i and its derivatives. The solutions are, therefore, additive, and the complete solution must contain four and only four arbitrary constants.

Instead of performing the elimination it is simpler and more instructive to solve by inspection by assuming

$$i_1 = A\epsilon^{kt}, \tag{13}$$

$$i_2 = B\epsilon^{kt}. \tag{14}$$

That these values are solutions is seen by a direct substitution of them in equations (11) and (12), giving

$$A\left\{L_1k + \frac{1}{C_1k}\right\} = MBk, \tag{15}$$

$$B\left\{L_2k + \frac{1}{C_2k}\right\} = MAk. \tag{16}$$

The product of these two equations gives

$$\left\{L_1k + \frac{1}{C_1k}\right\}\left\{L_2k + \frac{1}{C_2k}\right\} = M^2k^2, \tag{17}$$

which is independent of A and B

Equation (17) *is a relation that must be satisfied by* k, *in order that* (13) *and* (14) *may be a simultaneous system of values satisfying* (11) *and* (12).

85. Expression of (17) in Terms of Angular Velocities of the Separate Circuits.—Let us now write

$$\omega_1^2 = 1/L_1C_1, \tag{18}$$

$$\omega_2^2 = 1/L_2C_2. \tag{19}$$

It is seen that, since the resistances are negligible, ω_1 and ω_2 are the angular velocities of free oscillation of the two circuits of the system respectively, when each is alone and uninfluenced by the other. (Cf Arts. **8** and **15**).

If now we divide (17) by L_1kL_2k, we obtain

$$\left\{1 + \frac{\omega_1^2}{k^2}\right\}\left\{1 + \frac{\omega_2^2}{k^2}\right\} = \tau^2, \tag{20}$$

[1] The steps of this process are given in Art. **98** below.

where

$$\tau^2 = \frac{M^2}{L_1 L_2}. \tag{21}$$

The quantity τ is called the *coefficient of coupling* of the circuits.

Equation (20) may be solved as a quadratic in k^2. It is somewhat more direct to our purpose to solve (20) for the reciprocal of k rather than for k itself.

For this purpose, let us perform the indicated multiplication in (20) and divide the result by $\omega_1^2\omega_2^2$, obtaining

$$\frac{1}{k^4} + \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right)\frac{1}{k^2} = \frac{\tau^2 - 1}{\omega_1^2\omega_2^2}. \tag{22}$$

Completing the square and solving we obtain

$$\frac{1}{k} = \pm\sqrt{-\frac{1}{2}\left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)^2 + \frac{4\tau^2}{\omega_1^2\omega_2^2}}}. \tag{23}$$

Since τ, by the physics of the problem, is less than unity, it is seen that the quantity under the main radical is negative whether the plus or the minus be used before the second radical, since the original circuits are oscillatory. Whence, k is a pure imaginary quantity, and there are seen to be four different values of k consistent with (23).

These four values may be written

$$\left.\begin{aligned} k_1 &= j\omega' \text{ (say)}, & k_3 &= j\omega'' \text{(say)}, \\ k_2 &= -j\omega', & k_4 &= -j\omega'', \end{aligned}\right\} \tag{24}$$

where ω' and ω'' are given by the following relation, somewhat simplified from (23),

$$\frac{1}{\omega'} = +\sqrt{\frac{1}{2}\left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)^2 + \frac{4\tau^2}{\omega_1^2\omega_2^2}}} \tag{25}$$

$$\frac{1}{\omega''} = +\sqrt{\frac{1}{2}\left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)^2 + \frac{4\tau^2}{\omega_1^2\omega_2^2}}} \tag{26}$$

Taking the products of (25) and (26) and taking the reciprocal of the result, we find that

$$\omega'\omega'' = \frac{\omega_1\omega_2}{\sqrt{1 - \tau^2}} \tag{27}$$

which used as a multiplier of (25) and (26) gives

$$\omega' = \frac{\sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) - \frac{1}{2}\sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + 4\tau^2\omega_1^2\omega_2^2}}}{\sqrt{1 - \tau^2}} \tag{28}$$

$$\omega'' = \frac{\sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}\sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + 4\tau^2\omega_1^2\omega_2^2}}}{\sqrt{1 - \tau^2}} \tag{29}$$

In seeking for a solution of our original differential equations (11) and (12) we have now found four solutions, one corresponding to each value of k. These solutions are of the form of (13) and (14), and for each of the four solutions for i_1 we have a different arbitrary constant. Similarly for each of the four solutions for i_2 we have a separate arbitrary constant, but there are some relations among these constants.

Taking the sum of the four solutions for i_1 and likewise for i_2, we obtain

$$i_1 = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t} + A_3\epsilon^{k_3t} + A_4\epsilon^{k_4t} \tag{30}$$

$$i_2 = B_1\epsilon^{k_1t} + B_2\epsilon^{k_2t} + B_3\epsilon^{k_3t} + B_4\epsilon^{k_4t} \tag{31}$$

Equations (30) *and* (31) *are the complete solutions of the differential equations* (11) *and* (12). *In these solutions the several k's are given by* (24) *taken in connection with* (28) *and* (29). *The four A's and the four B's are arbitrary, except that each B is related to the corresponding A by a relation of the form of* (16) *and* (15). *The two relations* (16) *and* (15) *are not, however, independent since their product was used in determining the k's.*

86. Determination of the Periods of the Magnetically Coupled Pair of Resistanceless Circuits.—Let us leave for the present the question of the values of the arbitrary constants A and B, which are to be obtained from the initial conditions, and return to an examination of the k's, which may be used to give us the period or periods of the resulting oscillations that occur in the coupled system.

Since the k's are all imaginary quantities with the values given in (24), we may transform[1] the equations for i_1 and i_2 (namely, (30) and (31)) into the trigonometric forms

$$i_1 = I'_1 \sin(\omega't + \varphi'_1) + I''_1 \sin(\omega''t + \varphi''_1) \tag{32}$$

$$i_2 = I'_2 \sin(\omega't + \varphi'_2) + I''_2 \sin(\omega''t + \varphi''_2) \tag{33}$$

[1] Such a transformation is analyzed in Chapter IX, Art. **102.**

where the four I's and the four φ's are constants derivable from the A's and B's or from the initial conditions.

Fixing our attention upon the ω' and ω'', it is to be seen that both currents are doubly periodic, and that the two periods of the current i_1 in Circuit I are the same as the two periods of the current i_2 in the Circuit II. These two periods may be obtained from the corresponding angular velocities ω' and ω''.

Let these two periods be T' and T'', which are related to the corresponding angular velocities by the equations

$$T' = 2\pi/\omega' \tag{34}$$

$$T'' = 2\pi/\omega'' \tag{35}$$

Therefore, if we multiply equations (25) and (26) through by 2π, and recall that the periods T_1 and T_2 of the two circuits when alone are

$$T_1 = 2\pi/\omega_1 \tag{36}$$

$$T_2 = 2\pi/\omega_2 \tag{37}$$

we obtain

$$T' = +\sqrt{\frac{1}{2}\left(T_1^2 + T_2^2\right) + \frac{1}{2}\sqrt{\left(T_1^2 - T_2^2\right)^2 + 4\tau^2 T_1^2 T_2^2}} \tag{38}$$

$$T'' = +\sqrt{\frac{1}{2}\left(T_1^2 + T_2^2\right) - \frac{1}{2}\sqrt{\left(T_1^2 - T_2^2\right)^2 + 4\tau^2 T_1^2 T_2^2}} \tag{39}$$

These two equations may be written in a different form as follows:

$$T' = \frac{1}{2}\sqrt{T_1^2 + T_2^2 + 2T_1T_2\sqrt{1-\tau^2}} + \frac{1}{2}\sqrt{T_1^2 + T_2^2 - 2T_1T_2\sqrt{1-\tau^2}} \tag{40}$$

$$T'' = \frac{1}{2}\sqrt{T_1^2 + T_2^2 + 2T_1T_2\sqrt{1-\tau^2}} - \frac{1}{2}\sqrt{T_1^2 + T_2^2 - 2T_1T_2\sqrt{1-\tau^2}} \tag{41}$$

That (40) and (41) are respectively identical with (38) and (39) may be shown by squaring and extracting the square root of (40) and (41), by which operation we arrive at (38) and (39).

The equations (38) *and* (39), *or the alternative equations* (40) *and* (41), *give the two periods T' and T'' of the doubly periodic*

oscillation that occurs in the primary circuit of the coupled system. The same two periods occur also in the secondary circuit of the coupled system. These equations are exact only provided the resistances are negligible in their effects on the periods.

87. Determination of the Wavelengths of the Magnetically Coupled Pair of Resistanceless Circuits.—To obtain the resulting wavelengths in the coupled system, it is only necessary to multiply the periods by the velocity of light, and employ the relations

$$\left.\begin{aligned} \lambda' &= cT' \qquad & \lambda'' &= cT'' \\ \lambda_1 &= cT_1 \qquad & \lambda_2 &= cT_2 \end{aligned}\right\} \tag{42}$$

These values substituted into (38), (39), (40) and (41) give

$$\lambda' = \sqrt{\frac{1}{2}(\lambda_1^2 + \lambda_2^2) + \frac{1}{2}\sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 4\tau^2\lambda_1^2\lambda_2^2}} \tag{43}$$

$$\lambda'' = \sqrt{\frac{1}{2}(\lambda_1^2 + \lambda_2^2) - \frac{1}{2}\sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 4\tau^2\lambda_1^2\lambda_2^2}} \tag{44}$$

or the alternative results

$$\lambda' = \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\sqrt{1-\tau^2}} + \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\sqrt{1-\tau^2}} \tag{45}$$

$$\lambda'' = \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\sqrt{1-\tau^2}} - \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\sqrt{1-\tau^2}} \tag{46}$$

Equations (43) *and* (44), *or the alternative equations* (45) *and* (46), *give the two wavelengths* λ' *and* λ'' *of the doubly periodic oscillation that occurs in the primary circuit and also in the secondary circuit of the coupled system, provided the resistances are negligible in their effects on the resulting wavelengths.*[1]

88. Graphical Method of Finding λ' and λ''.—The equations given in the preceding section permit the calculation of λ' and λ'' when λ_1, λ_2, and τ are given.

When great accuracy is not required, the following graphical method may be employed. In Fig. 3, lay off AB equal to λ_1

[1] For experimental tests of these equations see PIERCE, *Physical Review*, 24, p. 152, 1907; also "Principles of Wireless Telegraphy," p. 228, McGraw-Hill, 1910.

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and BD also equal to λ_1 and in the same straight line with AB. At the point B draw the line BC making with BD an angle whose sine is τ. Make the length of BC equal to λ_2, then draw AC and DC. Call the lengths of AC and BC, b and a respectively. Then half the sum of b and a is the required wavelength λ', and half their difference is the required wavelength λ''.

This may be readily proved as follows:
Since

$$\sin\theta = \tau,$$

$$\cos\theta = \sqrt{1-\tau^2}.$$

By the geometrical proposition concerning the square of the side of a triangle opposite to an obtuse or an acute angle

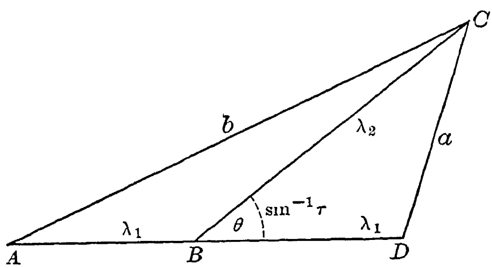

FIG 3—Showing geometrical construction for obtaining resultant wavelengths

$$\left.\begin{aligned} b^2 &= \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\cos\theta = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\sqrt{1-\tau^2} \\ a^2 &= \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\sqrt{1-\tau^2} \end{aligned}\right\} \quad (47)$$

whence, from (45) and (46)

$$\lambda' = \frac{b+a}{2}, \quad (48)$$

$$\lambda'' = \frac{b-a}{2}. \quad (49)$$

Exactly similar construction may be employed to give T' and T'', if all the λ's are replaced by the corresponding T's.

89. Simple Relations Among Wavelengths or Periods in a Magnetically Coupled Pair of Resistanceless Circuits.—By taking the sum of the squares of (38) and (39) and likewise the sum of the squares of (43) and (44), we obtain

$$T'^2 + T''^2 = T_1^2 + T_2^2, \quad (50)$$

$$\lambda'^2 + \lambda''^2 = \lambda_1^2 + \lambda_2^2. \quad (51)$$

Also, by taking the products of the same two pairs of equations, we obtain

$$T'T'' = T_1 T_2 \sqrt{1 - \tau^2} \tag{52}$$

$$\lambda'\lambda'' = \lambda_1 \lambda_2 \sqrt{1 - \tau^2} \tag{53}$$

90. Special Cases of the Coupled System of Negligible Resistances.—

Case I. Isochronism.—If the two circuits have the same period when each is alone,

$$\lambda_1 = \lambda_2 = \lambda \text{ (say)} \tag{54}$$

and

$$T_1 = T_2 = T \text{ (say)} \tag{55}$$

then equations (43), (44), (38), and (39) give

$$T' = T\sqrt{1+\tau} \qquad T'' = T\sqrt{1-\tau} \tag{56}$$

$$\lambda' = \lambda\sqrt{1+\tau} \qquad \lambda'' = \lambda\sqrt{1-\tau} \tag{57}$$

Case II. Negligible Coupling.—Whether the circuits are isochronous or not, if τ is sufficiently small so that terms involving it in (38) to (43) are negligible, these equations give

$$T' = T_1 \qquad T'' = T_2 \tag{58}$$

$$\lambda' = \lambda_1 \qquad \lambda'' = \lambda_2 \tag{59}$$

As to how small τ must be in order to be negligible depends upon the relative values of λ_1 and λ_2.

If $\lambda_1 = \lambda_2$, then by (57), to be negligible

$$\tau/2 < < 1 \tag{60}$$

where $< <$ means "is negligible in comparison with."

If, on the other hand, λ_1 and λ_2 are sufficiently different to make

$$4\tau^2\lambda_1^2\lambda_2^2 < \lambda_1^2 - \lambda_2^2 \tag{61}$$

we may expand the inner radical of (43) into

$$\frac{1}{2}(\lambda_1^2 - \lambda_2^2)\left\{1 + \frac{2\tau^2\lambda_1^2\lambda_2^2}{(\lambda_1^2 - \lambda_2^2)^2} + \ . \ . \ . \right\},$$

so that by (43)

$$\lambda' = \lambda_1\sqrt{1 + \frac{\tau^2\lambda_2^2}{(\lambda_1^2 - \lambda_2)^2} + \ . \ . \ .},$$

whence

$$\lambda' = \lambda_1,$$

provided

$$\frac{\tau^2}{2} << \frac{\lambda_1^2}{\lambda_2^2} - 1 \tag{62}$$

To decide whether or not the coefficient of coupling τ is negligible so as to permit the use of the simplified values of wavelength and period given in (58) *and* (59) *we first see if τ satisfies* (62). *If it does not, then we require that it must satisfy* (60) *in order to be negligible for the system of resistanceless circuits.*

The effects of the resistances on these relations will be given in a subsequent chapter.

CASE III. Perfect Coupling.—If the coefficient of coupling τ is equal to unity, the coupling is said to be perfect. Putting $\tau^2 = 1$ in (40), (41), (45), and (46), we obtain

$$T' = \sqrt{T_1^2 + T_2^2}, \qquad T'' = 0 \tag{63}$$

$$\lambda' = \sqrt{\lambda_1^2 + \lambda_2^2}, \qquad \lambda'' = 0 \tag{64}$$

CASE IV. Coupling Nearly Perfect.—Still assuming that the resistances of the two coupled circuits are zero, it is interesting to examine the values of the resulting wavelengths when τ is nearly equal to unity; that is, when the coupling is nearly perfect. To do this let

$$a = \frac{2\lambda_1\lambda_2\sqrt{1 - \tau^2}}{\lambda_1^2 + \lambda_2^2} \tag{65}$$

then (45) becomes

$$\lambda' = \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2}\left\{\sqrt{1 + a} + \sqrt{1 - a}\right\} \tag{66}$$

with a similar equation for λ''.

Expanding the square root terms by the binomial theorem, we obtain

$$\lambda' = \sqrt{\lambda_1^2 + \lambda_2^2} \tag{67}$$

$$\lambda'' = a\sqrt{\lambda_1^2 + \lambda_2^2}, \quad \text{provided} \tag{68}$$

$$a^2/8 << 1, \quad \text{where}$$

a has the value given in (65).

In the present chapter there have been laid down the fundamental differential equations for the free oscillation of two coupled circuits, and the differential equations in the special

case of negligible resistances in the circuits. General solutions of the resistanceless case have been obtained, and these solutions have been analyzed with reference to periods and wavelengths of the resultant oscillations.

In the next chapter, the discussion of the resistanceless case will be continued with special reference to the amplitudes of the oscillations under given initial conditions.

In later chapters analysis will be given in the cases where the resistances are not negligible.

CHAPTER VIII

THE FREE OSCILLATIONS OF TWO COUPLED RESISTANCELESS CIRCUITS. AMPLITUDES[1]

91. General Considerations.—In the determination of the amplitudes in the case of the free oscillations of two coupled resistanceless circuits, the result will depend upon the initial conditions assumed. Two sets of conditions will be taken, corresponding to I. Discharge of a Condenser in the Primary Circuit (Circuit I), II. Discharge of an Inductance. These will be given different major headings.

I. DISCHARGE OF A CONDENSER C_1

92. Determination of Amplitudes in Case of the Discharge of the Primary Condenser with Resistanceless Circuits.—Let the initial conditions in this particular case be that the conden-

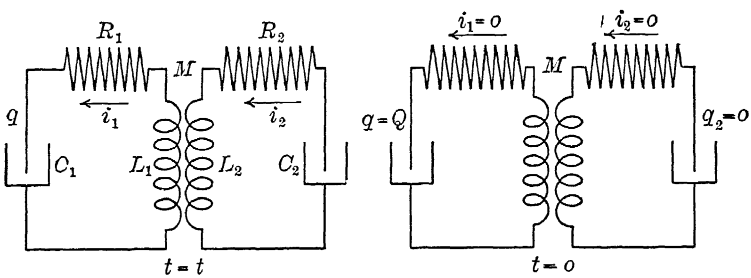

FIG. 1.—Two freely oscillating circuits. Right-hand diagram, state at $t = o$; left-hand diagram, state at any time t

ser C_1 in Fig. 1 is initially charged with a quantity of electricity Q and allowed to discharge.

Let us measure time from the instant at which the discharge begins. Then the initial conditions are

$$\text{When } t = 0, \quad \left.\begin{aligned} i_1 &= 0, \quad q_1 = Q = C_1E \\ i_2 &= 0, \quad q_2 = 0 \end{aligned}\right\} \tag{1}$$

[1] See references at beginning of Chapter VII, and also particularly E. Leon Chaffee, Amplitude Relations in Coupled Circuits, *Proc. Inst. Radio Engineers*, 4, p. 283, 1916. Professor Chaffee's paper contains also experimental verifications.

In Chapter VII, equations (30) and (31), we have found the general solutions for current in the form

$$i_1 = A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t} + A_3\epsilon^{k_3t} + A_4\epsilon^{k_4t} \tag{2}$$

$$i_2 = B_1\epsilon^{k_1t} + B_2\epsilon^{k_2t} + B_3\epsilon^{k_3t} + B_4\epsilon^{k_4t} \tag{3}$$

To the end that we may be able to introduce the initial condition in q_1 and q_2 we must obtain the equations for these quantities by integrating (2) and (3) with respect to t.

This integration gives

$$q_1 = \int i_1dt = \frac{A_1}{k_1}\epsilon^{k_1t} + \frac{A_2}{k_2}\epsilon^{k_2t} + \frac{A_3}{k_3}\epsilon^{k_3t} + \frac{A_4}{k_4}\epsilon^{k_4t} \tag{4}$$

$$q_2 = \int i_2dt = \frac{B_1}{k_1}\epsilon^{k_1t} + \frac{B_2}{k_2}\epsilon^{k_2t} + \frac{B_3}{k_3}\epsilon^{k_3t} + \frac{B_4}{k_4}\epsilon^{k_4t} \tag{5}$$

Now the several k's of these equations are known from Chapter VII, equation (24) to be in the resistanceless case

$$\left.\begin{aligned} k_1 &= j\omega', & k_3 &= j\omega'' \\ k_2 &= -j\omega', & k_4 &= -j\omega'' \end{aligned}\right\} \tag{6}$$

Furthermore, from Chapter VII, equation (16) we know that any B is related to the corresponding A by an equation of the form

$$\frac{B}{A} = \frac{Mk}{L_2k + \dfrac{1}{C_2k}} = \frac{M}{L_2}\frac{k^2}{k^2 + \omega_2^2} \tag{7}$$

in which the last term is obtained by replacing $1/L_2C_2$ by ω_2^2.

In using this equation we must give B, A, and k the same subscript. Doing this and replacing the subscripted k by its value from (6) we obtain the system of equations

$$\left.\begin{aligned} \frac{B_1}{A_1} = \frac{B_2}{A_2} = \frac{M}{L_2}\frac{\omega'^2}{\omega'^2 - \omega_2^2} = X \text{ (say)} \\ \frac{B_3}{A_3} = \frac{B_4}{A_4} = \frac{M}{L^2}\frac{\omega''^2}{\omega''^2 - \omega_2^2} = Y \text{ (say)} \end{aligned}\right\} \tag{8}$$

in which X and Y are abbreviations for the quantities set immediately before them in (8).

Now introducing our initial conditions (1) into (2), (3), (4) and (5) and making use of the equations (6) and (8), we obtain

$$0 = A_1 + A_2 + A_3 + A_4 \qquad \text{from (2)} \tag{9}$$

$$0 = X(A_1 + A_2) + Y(A_3 + A_4) \text{ from (3) and (8)} \qquad (10)$$

$$Q = \frac{A_1 - A_2}{j\omega'} + \frac{A_3 - A_4}{j\omega''} \qquad \text{from (4) and (6)} \qquad (11)$$

$$0 = \frac{X(A_1 - A_2)}{j\omega'} + \frac{Y(A_3 - A_4)}{j\omega''} \text{ from (5), (6) and (8)} \qquad (12)$$

Equations (9) and (10) give by elimination

$$(X - Y)\ (A_1 + A_2) = 0 \qquad (13)$$

$$(X - Y)\ (A_3 + A_4) = 0 \qquad (14)$$

while equations (11) and (12) give by elimination

$$-QY = (X - Y)\frac{A_1 - A_2}{j\omega'} \qquad (15)$$

$$QX = (X - Y)\frac{A_3 - A_4}{j\omega''} \qquad (16)$$

Unless $X = Y$, (13) and (14) give

$$A_1 = -A_2, \quad A_3 = -A_4 \qquad (17)$$

and these values substituted into (15) and (16) give

$$A_1 = -A_2 = \frac{-j\omega' QY}{2(X - Y)} \qquad (18)$$

and

$$A_3 = -A_4 = \frac{j\omega'' QX}{2(X - Y)} \qquad (19)$$

This derivation of the constants A_1, A_2, A_3 and A_4 is valid unless $X = Y$. By a comparison of (11) with (12) it is seen that X cannot equal Y unless both are zero. If both are zero, (8) shows that M is zero. If M is zero the Circuit II will have no current in it, and the Circuit I will be a single circuit with a condenser discharge in it satisfying the conditions given in Chapter II.

93. Periodic Equations for the Currents.—With these values of the A's and with proper values of the k's from (6) introduced into (2) we obtain

$$i_1 = \frac{-j\omega' QY}{X - Y}\,\frac{\epsilon^{j\omega' t} - \epsilon^{-j\omega' t}}{2} + \frac{j\omega'' QX}{X - Y}\,\frac{\epsilon^{j\omega'' t} - \epsilon^{-j\omega'' t}}{2} \qquad (20)$$

If we introduce j as a factor in the denominators of the exponential factors they become sines, and we have

$$i_1 = \frac{Q}{X - Y}\{\omega' Y \sin \omega' t - \omega'' X \sin \omega'' t\} \qquad (21)$$

In like manner using the values of the B's as given in (8), we obtain

$$i_2 = \frac{QXY}{X - Y}\{\omega' \sin \omega' t - \omega'' \sin \omega'' t\} \tag{22}$$

As a step toward replacing X and Y by their values, let us note from (8) that

$$X = \frac{M}{L_2}\frac{1}{1 - \omega_2^2/\omega'^2} = \frac{M}{L_2}\frac{1}{1 - T'^2/T_2^2}$$

whence

$$X = \frac{M}{L_2}\frac{T_2^2}{T_2^2 - T'^2} \tag{23}$$

In like manner from (8)

$$Y = \frac{M}{L_2}\frac{T_2^2}{T_2^2 - T''^2} \tag{24}$$

From these values of X and Y, we obtain

$$\frac{Y}{X - Y} = \frac{T_2^2 - T'^2}{T'^2 - T''^2} \tag{25}$$

$$\frac{X}{X - Y} = \frac{T_2^2 - T''^2}{T'^2 - T''^2} \tag{26}$$

Further, if we replace T' and T'' by their values from Chapter VII equations (38) and (39) we obtain

$$\frac{Y}{X - Y} = \frac{1}{2}\left\{-1 - \frac{1}{\sqrt{1 + \dfrac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}}\right\} \tag{27}$$

$$\frac{X}{X - Y} = \frac{1}{2}\left\{+1 - \frac{1}{\sqrt{1 + \dfrac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}}\right\} \tag{28}$$

Introducing these values into (21) we obtain

$$i_1 = \frac{-Q}{2}\left[\left\{\frac{1}{\sqrt{1 + \dfrac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}} + 1\right\}\omega' \sin \omega' t + \left\{\frac{-1}{\sqrt{1 + \dfrac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}} + 1\right\}\omega'' \sin \omega'' t\right] \tag{29}$$

Let us next determine i_2, which can be done by multiplying the equation (23) by (25) obtaining

$$\frac{XY}{X - Y} = \frac{M}{L_2}\frac{T_2^2}{T'^2 - T''^2} \tag{30}$$

which by (38) and (39) Chapter VII

$$= \frac{M}{L_2} \frac{T_2^2}{\sqrt{(T_1^2 - T_2^2)^2 + 4\tau^2 T_1^2 T_2^2}}.$$

This introduced into (22) gives

$$i_2 = \frac{MQ}{L_2} \frac{T_2^2}{\sqrt{(T_1^2 - T_2^2)^2 + 4\tau^2 T_1^2 T_2^2}} \{\omega' \sin \omega' t - \omega'' \sin \omega'' t\} \quad (31)$$

Equations (29) and (31) give the complete expressions for the currents in the two circuits of the coupled system having negligible resistances and excited by discharging at the time $t = 0$ the condenser C_1 with an initial charge Q.

94. Relative Amplitudes of Current in the Coupled System of Negligible Resistances Excited by a Condenser Discharge.—If we write the equations for i_1 and i_2 respectively in the form

$$i_1 = I'_1 \sin \omega' t + I''_1 \sin \omega'' t \quad (32)$$

$$i_2 = I'_2 \sin \omega' t + I''_2 \sin \omega'' t \quad (33)$$

it is seen that the ratios of amplitudes for the same frequency in the different circuits may be written. [See (21) and (22).]

$$\frac{I'_2}{I'_1} = X = \frac{M}{L_2} \frac{T_2^2}{T_2^2 - T'^2} = \tau \frac{\sqrt{C_2}}{\sqrt{C_1}} \frac{T_1 T_2}{T_2^2 - T'^2} \quad (34)$$

$$\frac{I''_2}{I''_1} = Y = \frac{M}{L_2} \frac{T_2^2}{T_2^2 - T''^2} = \tau \frac{\sqrt{C_2}}{\sqrt{C_1}} \frac{T_1 T_2}{T_2^2 - T''^2} \quad (35)$$

Also it is seen that the ratio of amplitudes of the two different frequencies in the same circuit are for primary and secondary respectively

$$\frac{I''_1}{I'_1} = \frac{-\omega'' X}{\omega' Y} = \frac{-T' X}{T'' Y} = \frac{-T'(T_2^2 - T''^2)}{T''(T_2^2 - T'^2)} \quad (36)$$

$$\frac{I''_2}{I'_2} = -\frac{\omega''}{\omega'} = -\frac{T'}{T''} \quad (37)$$

Equation (34) gives the ratio of amplitude of current in the secondary circuit to that in the primary circuit for the period T'. Equation (35) gives a similar ratio of amplitudes for the period T''. Equation (36) gives the ratio of amplitude of current of period T'' to the amplitude of current of period T' in the same (primary) circuit. Equation (37) is a similar ratio for the sec-

ondary circuit. These equations hold for excitation by the discharge of the primary condenser with resistanceless circuits.

II. DISCHARGE OF AN INDUCTANCE

95. Determination of Amplitudes when the Coupled System of Negligible Resistances is Excited by the Discharge of the Primary Inductance.—Let us now determine the solution of the resistanceless coupled circuit problem when the excitation is produced by sending a steady current through the inductance of Circuit I, and then isolating it as was done in the buzzer excitation process of Chapter II.

The differential equations are the same as in the problem already treated and give therefore the same frequencies as before. The amplitudes, however, which are determined by the initial conditions will now be different from those of the previous sections.

If we measure time from the instant of isolating the current in the primary inductance, the initial conditions are as follows:

$$\text{When } t = 0,\ i_1 = I,\ i_2 = 0,$$
$$q_1 = 0,\ q_2 = 0 \tag{38}$$

By comparison with the equations (9) to (12) it will be seen that these initial conditions require

$$I = A_1 + A_2 + A_3 + A_4 \tag{39}$$

$$0 = X(A_1 + A_2) + Y(A_3 + A_4) \tag{40}$$

$$0 = \frac{A_1 - A_2}{j\omega'} + \frac{A_2 - A_3}{j\omega''} \tag{41}$$

$$0 = \frac{X(A_1 - A_2)}{j\omega'} + \frac{Y(A_3 - A_4)}{j\omega''} \tag{42}$$

Elimination among these equations gives

$$A_1 = A_2 = \frac{YI}{2(Y - X)} \tag{43}$$

$$A_3 = A_4 = \frac{XI}{2(X - Y)} \tag{44}$$

The several B's have the same ratio to the corresponding A's as in the condenser discharge problem.

These constants substituted into equations (2) and (3) give, after ' . ' ' : ' ' transformation as before, the results

$$i_1 = \frac{-I}{X - Y}\{Y \cos \omega' t - X \cos \omega'' t\} \tag{45}$$

$$i_2 = \frac{-IXY}{X - Y}\{\cos \omega' t - \cos \omega'' t\} \tag{46}$$

By comparison of these equations for current in this case of inductance excitation with the corresponding equations for current in the previous problem of capacity excitation, it will be seen that equations (45) and (46) take the form

$$i_1 = \frac{I}{2}\left[\left\{1 + \frac{1}{\sqrt{1 + \frac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}}\right\}\cos \omega' t + \left\{1 - \frac{1}{\sqrt{1 + \frac{4\tau^2 T_1^2 T_2^2}{(T_1^2 - T_2^2)^2}}}\right\}\cos \omega'' t\right] \tag{47}$$

$$i_2 = -\frac{MI}{L_2}\frac{T_2^2}{\sqrt{(T_1^2 - T_2^2)^2 + 4\tau^2 T_1^2 T_2^2}}[\cos \omega' t - \cos \omega'' t] \tag{48}$$

Equations (47) and (48) give respectively the primary and secondary current in a coupled system of two circuits of negligible resistances, excited by sending a steady current I through the inductance of the primary circuit and isolating it at a time $t = 0$.

96. Relative Amplitudes of Current in the Resistanceless Coupled System Excited by Isolating a Current in the Primary Circuit.—If now in this case we write the expressions for the currents in the abbreviated forms

$$i_1 = J'_1 \cos \omega' t + J''_1 \cos \omega'' t \tag{49}$$

$$i_2 = J'_2 \cos \omega' t + J''_2 \cos \omega'' t \tag{50}$$

and compare the amplitudes we have

$$\frac{J'_2}{J'_1} = X = \tau\frac{\sqrt{C_2}}{\sqrt{C_1}}\frac{T_1 T_2}{T_2^2 - T'^2} \tag{51}$$

$$\frac{J''_2}{J''_1} = Y = \tau\frac{\sqrt{C_2}}{\sqrt{C_1}}\frac{T_1 T_2}{T_2^2 - T''^2} \tag{52}$$

$$\frac{J''_1}{J'_1} = -\frac{X}{Y} = -\frac{T_2^2 - T''^2}{T_2^2 - T'^2} \tag{53}$$

$$\frac{J''_2}{J'_2} = -1 \tag{54}$$

Equations (51) *to* (54) *give the relative amplitudes of current in the resistanceless coupled system of two circuits excited by the discharge of an inductance in the primary circuit. The discharge is produced by isolating a constant current I in the primary inductance at* $t = 0$.

It is to be noted that two of the ratios (51) *and* (52) *are the same as in the case of the condenser excitation, and two of the ratios* (53) *and* (54) *are different from the case of condenser excitation.*

It is also to be noted that cosines appear in the present case, where sines appeared in the case of the other form of excitation.

CHAPTER IX

THE FREE OSCILLATION OF TWO INDUCTIVELY COUPLED CIRCUITS. PERIODS AND DECREMENTS. TREATMENT WITHOUT NEGLECTING RESISTANCES[1]

97. Differential Equations.—It is proposed to treat in the present chapter the theory of the free oscillation of two coupled circuits such as are shown diagrammatically in Fig. 1. The method is similar to that employed in Chapters VII and VIII, except that now the resistances are to be retained wherever their values are significant.

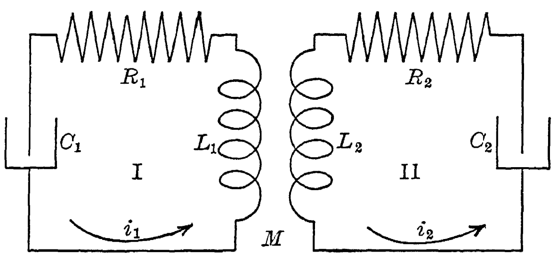

FIG. 1.—Diagram of circuits.

The differential equations are those given in equations (1) and (2) of Chapter VII, which are here rewritten with all the terms transposed to the left-hand side; namely,

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{\int i_1 dt}{C_1} - M \frac{di_2}{dt} = 0 \qquad (1)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{\int i_2 dt}{C_2} - M \frac{di_1}{dt} = 0 \qquad (2)$$

98. Elimination to Show that the Resulting Equations are of the Fourth Order.—Let us eliminate i_2 from the two equations and show that the resulting equation in i_1 is a differential equation of the fourth order.

[1] See references at beginning of Chapters VII and VIII. The present treatment is more complete than the treatment in the references.

First add M times equation (2) to L_2 times equation (1), and differentiate, obtaining

$$(L_1L_2 - M^2)\frac{d^2i_1}{dt_2} + L_2R_1\frac{di_1}{dt} + \frac{L_2i_1}{C_1} + R_2M\frac{di_2}{dt} + \frac{Mi_2}{C_2} = 0 \quad (3)$$

Add R_2 times (1) to (3) and differentiate, obtaining

$$(L_1L_2 - M^2)\frac{d^3i_1}{dt^3} + (L_1R_2 + L_2R_1)\frac{d^2i_1}{dt^2} + (\frac{L_2}{C_1} + R_1R_2)\frac{di_1}{dt} + \frac{R_2i_1}{C_1} + \frac{M}{C_2}\frac{di_2}{dt} = 0 \quad (4)$$

Add $1/C_2$ times (1) to (4), and differentiate, obtaining

$$(L_1L_2 - M^2)\frac{d^4i_1}{dt^4} + (L_1R_2 + L_2R_1)\frac{d^3i_1}{dt^3} + (\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1R_2)\frac{d^2i_1}{dt^2} + (\frac{R_1}{C_2} + \frac{R_2}{C_1})\frac{di_1}{dt} + \frac{i_1}{C_1C_2} = 0 \quad (5)$$

In the same way the elimination of i_1 instead of i_2 gives for i_2 the same equation except that i_2 is substituted for i_1

Equation (5) is a homogeneous linear differential equation of the fourth order. The complete solution has four arbitrary constants, and any solution that has four arbitrary constants is complete.

Instead of proceeding directly to a solution of (5) by introducing an exponential with t in the exponent, it is somewhat more convenient to make our substitutions in (1) and (2) as was done in Chapter VII. We shall make no use of (5) further than to note that the complete integral has four arbitrary constants.

99. First Step in the Solution of (1) and (2).—Let us begin the treatment of the pair of simultaneous equations (1) and (2) by letting

$$i_1 = A\epsilon^{kt}, \quad i_2 = B\epsilon^{kt} \quad (6)$$

These values, substituted into (1) and (2), give, after division by ϵ^{kt},

$$A(L_1k + R_1 + \frac{1}{C_1k}) = MBk \quad (7)$$

and

$$B(L_2k + R_2 + \frac{1}{C_2k}) = MAk \quad (8)$$

Taking the product of (7) and (8), we obtain

$$(L_1k + R_1 + \frac{1}{C_1k})(L_2k + R_2 + \frac{1}{C_2k}) = M^2k^2 \quad (9)$$

Dividing (9) by $L_1L_2k^2$, we obtain, in terms of abbreviations next given, the equation

$$\left(1 + \frac{2a_1}{k} + \frac{\Omega_1^2}{k^2}\right)\left(1 + \frac{2a_2}{k} + \frac{\Omega_2^2}{k^2}\right) = \tau^2 \tag{10}$$

where, as in previous chapters,

$$\tau = \frac{M}{\sqrt{L_1L_2}} \tag{11}$$

$$a_1 = \frac{R_1}{2L_1}, \qquad a_2 = \frac{R_2}{2L_2} \tag{12}$$

$$\Omega_1^2 = \frac{1}{L_1C_1}, \quad \Omega_2^2 = \frac{1}{L_2C_2} \tag{13}$$

Among these abbreviations note that the quantities Ω_1 and Ω_2 are related to the corresponding ω and a by the equations

$$\Omega_1^2 = \omega_1^2 + a_1^2, \quad \Omega_2^2 = \omega_2^2 + a_2^2 \tag{14}$$

as may be seen by reference to (viii) at the beginning of Chapter II.

Equation (10) is an equation of the fourth degree that k must satisfy, in order for (6) to be solutions of the original differential equations. In (10) the quantity τ, defined by (11) is called the coefficient of coupling of the circuits. The quantities a_1 and a_2 are the logarithmic decrements per second, or damping constants, of the separate circuits when each is alone and uninfluenced by the other. Ω_1 and Ω_2 are the undamped angular velocities of the two circuits respectively when they are uninfluenced by each other.

In equation (14) ω_1 and ω_2 are the free angular velocities of the separate circuits. It is seen that the undamped angular velocities Ω_1 and Ω_2 are equal to the free angular velocities in those cases in which $a_1^2/2\omega_1^2$ and $a_2^2/2\omega_2^2$ are negligible in comparison with unity.

100. Note on the Constants A and B.—Returning now to equation (10), let us designate the four k's that are roots of (10) by k_1, k_2, k_3, and k_4. Then by (6) for each of the k's there will be a corresponding A and B, to which we shall give subscripts 1, 2, 3, and 4 identical with the respective subscripts of k, obtaining

$$i_1 = A_n\epsilon^{k_nt}, \quad i_2 = B_n\epsilon^{k_nt},$$

where

$$n = 1, 2, 3, 4.$$

Applying to these solutions the principle of additivity, we shall have as the complete integral of the differential equations (1) and (2) the following

$$\begin{aligned} i_1 &= A_1\epsilon^{k_1t} + A_2\epsilon^{k_2t} + A_3\epsilon^{k_3t} + A_4\epsilon^{k_4t} \\ &= \Sigma A_n\epsilon^{k_nt} \end{aligned} \tag{15}$$

and likewise

$$i_2 = \Sigma B_n\epsilon^{k_nt}, \tag{16}$$

where $\quad n = 1, 2, 3, 4$

The constants A_n and B_n are arbitrary constants of integration. Although there are eight of these constants only the four A's are independent of each other, for each B is related to the corresponding A by an equation of the form of (7) or (8), in which we must give A and B either of the common subscripts 1, 2, 3, 4. Calling any one of these common subscripts by the generic designation n, we have from (7) and (8)

$$A_n\left(L_1k_n + R_1 + \frac{1}{C_1k_n}\right) = MB_nk_n \tag{17}$$

$$B_n\left(L_2k_n + R_2 + \frac{1}{C_2k_n}\right) = MA_nk_n \tag{18}$$

Either of the relations (17) or (18) may be used to determine B_n from A_n, but if both (17) and (18) are used they give no more restriction than one alone, for the two equations are not independent, as their product has been used in determining k_n.

The eight arbitrary constants are thus reduced to four by having four relations among them. These four relations are obtained by giving n successively the values 1, 2, 3, and 4.

The four arbitrary constants to which the eight are reduced are to be determined by the initial conditions in any specific problem. We shall postpone the determination of these constants A_n and B_n to the next Chapter, and shall proceed in this chapter to a discussion of the values of k_1, k_2, k_3, and k_4, which are the roots of the fourth degree equation (10).

101. Expression of the Roots k as Complex Quantities and the Currents as Periodic Functions of the Time.—Expanding (10) by multiplying the factors together, we obtain

$$k^4 + \frac{2(a_1 + a_2)k^3}{1 - \tau^2} + \frac{(\Omega_1^2 + \Omega_2^2 + 4a_1a_2)k^2}{1 - \tau^2} + \frac{2(a_1\Omega_2^2 + a_2\Omega_1^2)k}{1 - \tau^2} + \frac{\Omega_1^2\Omega_2^2}{1 - \tau^2} = 0 \tag{19}$$

Let us now write the four values of k that are the roots of (19) in the complex forms

$$\left.\begin{aligned} k_1 &= -a' + j\omega', & k_3 &= -a'' + j\omega'' \\ k_2 &= -a' - j\omega', & k_4 &= -a'' - j\omega'' \end{aligned}\right\} \qquad (20)$$

They can be written in this form for if any root is a complex quantity, the conjugate complex is also a root. Real roots, if they exist, must therefore be two or four in number. To cover this contingency of real roots it is only necessary to make ω' or ω'', or both, imaginary. The a's always remain real.

With the use of these complex roots, equations (15) and (16) can be transformed into

$$i_1 = I'_1\epsilon^{-a't} \sin (\omega't + \varphi'_1) + I''_1\epsilon^{-a''t} \sin (\omega''t + \varphi''_1) \quad (21)$$

$$i_2 = I'_2\epsilon^{-a't} \sin (\omega't + \varphi'_2) + I''_2\epsilon^{-a''t} \sin (\omega''t + \varphi''_2) \quad (22)$$

as is proved in the next section.

102. Digression to Prove Validity of the Transformation of (15) into (21).—With the values of k_1, k_2, k_3, and k_4 given in (20) we have

$$\epsilon^{k_1t} = \epsilon^{-a't}\,\epsilon^{j\omega't} = \epsilon^{-a't}(\cos \omega't + j \sin \omega't),$$

$$\epsilon^{k_2t} = \epsilon^{-a't}\,\epsilon^{-j\omega't} = \epsilon^{-a't}(\cos \omega't - j \sin \omega't),$$

etc., so that (15) can be written

$$i_1 = \epsilon^{-a't}\{(A_1 + A_2) \cos \omega't + j(A_1 - A_2) \sin \omega't\}$$
$$+ \epsilon^{-a''t}\{(A_3 + A_4) \cos \omega''t + j(A_3 - A_4) \sin \omega''t\} \quad (23)$$

Therefore,

$$i_1 = I'_1\epsilon^{-a't}\sin (\omega't + \varphi'_1) + I''_1\epsilon^{-a''t} \sin (\omega''t + \varphi''_1) \quad (24)$$

provided

$$\left.\begin{aligned} A_1 + A_2 &= I'_1 \sin \varphi'_1, & j(A_1 - A_2) &= I'_1 \cos \varphi'_1 \\ A_3 + A_4 &= I''_1 \sin \varphi''_1, & j(A_3 - A_4) &= I''_1 \cos \varphi''_1 \end{aligned}\right\} \quad (25)$$

In order for (25) to be satisfied by real values of I'_1 and φ'_1, it is seen that $A_1 + A_2$ must be real and $A_1 - A_2$ must be imaginary; that is to say, A_1 and A_2 must be in general conjugate complexes. This looks like an additional restriction on the arbitrariness of A_1 and A_2 that we have imposed by the transformation. But, as a matter of fact, this limitation is imposed by the equations (15)

if we require that the current i_1 be real and if we assume that ω' is real, for this assumption gives at once (23) that requires the conjugate relation of A_1 and A_2.

If on the other hand ω' is imaginary, let

$$\omega' = -j\omega_h, \text{ where } \omega_h \text{ is real, then}$$

$$\cos \omega' = \cosh \omega_h, \text{ and } j \sin \omega' = -\sinh \omega_h,$$

so that, in this case, (23) shows that both A_1 and A_2 are reals. With these two A's real, (25) shows that both I'_1 and φ'_1 are imaginary. This is still consistent with (24), for if ω', φ'_1 and I'_1 are all imaginary, the first term of the right-hand side of (24) remains real.

It is thus seen that the transformation of (15) into (21) is correct algebraically and that it does not put any additional restriction on i_1.

103. Angular Velocities and Damping Constants. Double Periodicity.—Returning to equations (21) and (22), it is seen that, if ω' and ω'' are real quantities, the primary current i_1 and the secondary current i_2 is each doubly periodic, with the two angular velocities ω' and ω'', and that each of the oscillations has its own damping constant, a' for ω' and a'' for ω''.

Both circuits have the same two angular velocities ω' and ω'', and both have the same damping constants a' and a''.

104. Relations Among the Damping Constants, the Angular Velocities and the Constants of the Circuits.—We shall now make use of the following propositions proved in treatises on the Theory of Algebraic Equations:

If k_1, k_2, k_3, and k_4 are the four roots of the fourth degree algebraic equation (19), then

I. The sum of the four values of the roots is equal to the negative of the coefficient of k^3 in (19),

II. The sum of the products of the roots taken two and two equals the coefficient of k^2,

III. The sum of the products of the roots taken three at a time is equal to the negative of the coefficient of k,

IV. The product of the four roots is equal to the term of (19) not involving k.

By direct computation, using the form (20) of the roots, we obtain the following relations:

$$a' + a'' = \frac{a_1 + a_2}{1 - \tau^2} \tag{26}$$

$$\Omega'^2 + \Omega''^2 + 4a'a'' = \frac{\Omega_1^2 + \Omega_2^2 + 4a_1a_2}{1 - \tau^2} \tag{27}$$

$$a'\Omega''^2 + a''\Omega'^2 = \frac{a_1\Omega_2^2 + a_2\Omega_1^2}{1 - \tau^2} \tag{28}$$

$$\Omega'^2\Omega''^2 = \frac{\Omega_1^2\Omega_2^2}{1 - \tau^2} \tag{29}$$

in which

$$\Omega'^2 = a'^2 + \omega'^2 \tag{30}$$

$$\Omega''^2 = a''^2 + \omega''^2 \tag{31}$$

The definitions of the other quantities are given in (11), (12) and (13).

The equations (26) *to* (29) *are the exact relations that the primed quantities, regarded as unknown, bear to the subscripted quantities, regarded as known.*

The primed quantities are the resultant damping constants and angular velocities in the coupled system, while the subscripted quantities are quantities belonging to the circuits I and II respectively when each is standing alone and uninfluenced by the other.

The problem of finding the damping constants and angular velocities in the coupled system consists in elimination among these equations in such a manner as to obtain each of the primed quantities in an equation not involving the other primed quantities The equations are sufficient in number for this purpose, and the eliminations, though difficult, are effected in the sections that follow.

105. Introduction of Undamped Periods in Place of Undamped Angular Velocities.—It is proposed now to modify the relations (26) to (29) by introducing periods in the place of angular velocities.

Let

$$\left.\begin{aligned} T_1 &= 2\pi/\omega_1, & T' &= 2\pi/\omega' \\ T_2 &= 2\pi/\omega_2, & T'' &= 2\pi/\omega'' \end{aligned}\right\} \tag{32}$$

and

$$\left.\begin{aligned} S_1 &= 2\pi/\Omega_1, & S' &= 2\pi/\Omega' \\ S_2 &= 2\pi/\Omega_2, & S'' &= 2\pi/\Omega'' \end{aligned}\right\} \tag{33}$$

Here T_1 and T_2 are the periods of the two circuits, respectively, when not coupled; T' and T'' are the periods that coexist in each

circuit when coupled; while the corresponding S's are the several undamped periods.

It is often true that the S's are close approximations to the T's in single oscillatory circuits, but when the circuits are coupled the arithmetical differences between the various S's or T's appear in the equations, and in those cases it is not safe to replace the S's by T's without special investigation.

Returning now to our coefficient equations (26) to (29), let us divide (26), (27), and (28) each by (29) and multiply by $(2\pi)^2$ or $(2\pi)^4$, as required, obtaining, respectively,

$$(a' + a'')S'^2S''^2 = (a_1 + a_2)S_1^2S_2^2 \tag{34}$$

$$S'^2 + S''^2 = S_1^2 + S_2^2 + zS_1S_2 \text{ (say)} \tag{35}$$

$$a'S'^2 + a''S''^2 = a_1S_1^2 + a_2S_2^2 \tag{36}$$

$$S'^2S''^2 = S_1^2S_2^2(1 - \tau^2) \tag{37}$$

The z that occurs in (35) has the value

$$z = \left\{a_1a_2 - a'a''(1 - \tau^2)\right\}\frac{S_1S_2}{\pi^2} \tag{38}$$

These equations written in terms of undamped periods are the equivalents of (26) *to* (29), *which were obtained directly from the coefficients of the fourth degree equation* (19). *They are exact.*

The quantity z, as defined in (38), *will be left undetermined in the first stages of the eliminations, but will finally be expressed in terms of known quantities.*

106. Combination for Undamped Periods.—We shall now form certain combinations of the equations (34) to (38). The first combination is here designated *combination for undamped periods.*

Let us add twice the square root of (37) to (35) and extract the square root; and then let us subtract twice the square root of (37) from (35) and extract the square root. By these operations we obtain, respectively,

$$S' + S'' = \sqrt{S_1^2 + S_2^2 + zS_1S_2 + 2S_1S_2\sqrt{1 - \tau^2}} \tag{39}$$

$$S' - S'' = \sqrt{S_1^2 + S_2^2 + zS_1S_2 - 2S_1S_2\sqrt{1 - \tau^2}} \tag{40}$$

In choosing the positive sign before the main radical in (40) *we are specifying that of the two quantities S' and S'' we shall call the greater S'.*

Now taking respectively the sum and difference of these two equations and dividing by 2, we obtain

$$\begin{aligned} S' = {} & \frac{1}{2}\sqrt{S_1^2 + S_2^2 + zS_1S_2 + 2S_1S_2\sqrt{1 - \tau^2}} \\ & + \frac{1}{2}\sqrt{S_1^2 + S_2^2 + zS_1S_2 - 2S_1S_2\sqrt{1 - \tau^2}} \end{aligned} \tag{41}$$

$$\begin{aligned} S'' = {} & \frac{1}{2}\sqrt{S_1^2 + S_2^2 + zS_1S_2 + 2S_1S_2\sqrt{1 - \tau^2}} \\ & - \frac{1}{2}\sqrt{S_1^2 + S_2^2 + zS_1S_2 - 2S_1S_2\sqrt{1 - \tau^2}} \end{aligned} \tag{42}$$

Equations (41) and (42) are the values of the undamped periods in the coupled system. They are exact. It will be noticed, however, that the expressions involve z and hence a' and a''. We shall later show how to obtain z in terms of known quantities.

107. Combination for Damping Relations.—Returning now to the equations (34) to (37) let us, first, subtract $1/S'^2$ times (34) from (36); second, subtract $1/S''^2$ times (34) from (36). Dividing the differences obtained by $S'^2 - S''^2$, we have

$$a' = \frac{a_1S_1^2 + a_2S_2^2 - (a_1 + a_2)\dfrac{S_1^2S_2^2}{S'^2}}{S'^2 - S''^2} \tag{43}$$

$$a'' = -\frac{a_1S_1^2 + a_2S_2^2 - (a_1 + a_2)\dfrac{S_1^2S_2^2}{S''^2}}{S'^2 - S''^2} \tag{44}$$

Expanding these equations by replacing S'^2 and S''^2 by their values from (41) and (42), we obtain

$$a' = \frac{a_1 + a_2}{2(1 - \tau^2)} - \frac{\dfrac{a_1 + a_2}{2(1 - \tau^2)}(S_1^2 + S_2^2 + zS_1S_2) - (a_1S_1^2 + a_2S_2^2)}{\sqrt{(S_1^2 + S_2^2 + zS_1S_2)^2 - 4S_1^2S_2^2\,(1 - \tau^2)}} \tag{45}$$

$$a'' = \frac{a_1 + a_2}{2(1 - \tau^2)} + \frac{\dfrac{a_1 + a_2}{2(1 - \tau^2)}(S_1^2 + S_2^2 + zS_1S_2) - (a_1S_1^2 + a_2S_2^2)}{\sqrt{(S_1^2 + S_2^2 + zS_1S_2)^2 - 4S_1^2S_2^2\,(1 - \tau^2)}} \tag{46}$$

Equations (43) and (44), or the alternative equations (45) and (46), are exact relations for the damping constants a' and a'' of the two oscillations in the coupled system. It will be noted, however,

that z involves a' and a'', so that these quantities have not yet been completely isolated.

Before entering upon a determination of z, let us write out still another form of expression for the damping constants, obtained directly from the definition (38) of z, which by transposition gives

$$a'\,a'' = \frac{a_1 a_2}{1-\tau^2} - \frac{\pi^2 z}{S_1 S_2\,(1-\tau^2)}.$$

We have also from (34) and (37) by division

$$a' + a'' = \frac{a_1 + a_2}{1-\tau^2}.$$

Now taking four times the first of these equations from the square of the second, and extracting the square root, we obtain

$$a' - a'' = \pm\sqrt{\frac{(a_1+a_2)^2}{(1-\tau^2)^2} - \frac{4a_1a_2}{1-\tau^2} + \frac{4\pi^2 z}{S_1S_2\,(1-\tau^2)}}.$$

In order to determine which sign to use before this radical it is necessary to determine from an independent examination of (45) and (46) whether a' is greater than or less than a''. It will be noted that if

$$\frac{(a_1+a_2)\,(S_1^2+S_2^2+zS_1S_2)}{2\,(1-\tau^2)} > a_1S_1^2 + a_2S_2^2 \qquad (47)$$

then $a' < a''$, and we must use the minus sign before the radical above. Under this condition, elimination between the equation for $a' + a''$ and that for $a' - a''$ gives

$$a' = \frac{1}{2}\,\frac{a_1+a_2}{1-\tau^2} - \frac{1}{2}\sqrt{\frac{(a_1+a_2)^2}{(1-\tau^2)^2} - \frac{4a_1a_2}{1-\tau^2} + \frac{4\pi^2 z}{S_1S_2\,(1-\tau^2)}} \qquad (48)$$

$$a'' = \frac{1}{2}\,\frac{a_1+a_2}{1-\tau^2} + \frac{1}{2}\sqrt{\frac{(a_1+a_2)^2}{(1-\tau^2)^2} - \frac{4a_1a_2}{1-\tau^2} + \frac{4\pi^2 z}{S_1S_2\,(1-\tau^2)}} \qquad (49)$$

In using these equations (48) and (49) it is to be especially noted that if the inequality (47) is not fulfilled the signs before the radicals in (48) and (49) are to be interchanged. This rule of signs is based also on the stipulation that of the two quantities S' and S'' the greater is designated S'.

Having now obtained a variety of expressions for the determination of S', S'', a', and a'', we shall next obtain an explicit equation for z in terms of known quantities.

108. Equation for z in Terms of Known Quantities.—We shall now obtain an equation for z in terms of known quantities.

Since z involves the product of a' and a'', let us form this product by multiplying (45) by (46). In performing this multiplication we shall use the temporary abbreviation

$$D = (S_1^2 + S_2^2 + zS_1S_2)^2 - 4S_1^2S_2^2(1 - \tau^2),$$

which expanded gives

$$D = (S_1^2 - S_2^2)^2 + 2S_1S_2z(S_1^2 + S_2^2) + z^2S_1^2S_2^2 + 4\tau^2S_1^2S_2^2 \quad (50)$$

Proceeding now to take the product of (45) and (46), and clearing the result of fractions, we obtain

$$\begin{aligned} a'a''D(1 - \tau^2) = & -(a_1S_1^2 + a_2S_2^2)^2 + \tau^2(a_1S_1^2 + a_2S_2^2)^2 \\ & + (a_1 + a_2)\ (a_1S_1^2 + a_2S_2^2)\ (S_1^2 + S_2^2) \\ & - (a_1 + a_2)^2S_1^2S_2^2 + zS_1S_2(a_1 + a_2) \\ & \qquad (a_1S_1^2 + a_2S_2^2), \end{aligned}$$

whence

$$\begin{aligned} a'a''D(1 - \tau^2) = & \ \tau^2(a_1S_1^2 + a_2S_2^2)^2 \\ & + zS_1S_2(a_1^2S_1^2 + a_1a_2S_2^2 + a_1a_2S_1^2 + a_2^2S_2^2) \\ & + a_1a_2(S_1^4 + S_2^4 - 2S_1^2S_2^2). \end{aligned}$$

Now let us subtract a_1a_2D from the left-hand side of this equation, and from the right-hand side this same quantity with D replaced by its value from (50), and note that the difference obtained for the left-hand side is $-z\pi^2D/S_1S_2$ by (38). We thus obtain

$$\frac{-z\pi^2D}{S_1S_2} = \tau^2(a_1S_1^2 - a_2S_2^2)^2 + zS_1S_2\,(a_1 - a_2)\,(a_1S_1^2 - a_2S_2^2) - z^2a_1a_2S_1^2S_2^2.$$

Replacing D by its value from (50) and collecting terms we obtain

$$\begin{aligned} z^3 & + z^2\left\{\frac{2(S_1^2 + S_2^2)}{S_1S_2} - \frac{a_1a_2S_1S_2}{\pi^2}\right\} \\ & + z\left\{\frac{(S_1^2 - S_2^2)^2}{S_1^2S_2^2} + 4\tau^2 + \frac{(a_1 - a_2)\,(a_1S_1^2 - a_2S_2^2)}{\pi^2}\right\} \\ & + \frac{\tau^2(a_1S_1^2 - a_2S_2^2)^2}{\pi^2S_1S_2} = 0. \end{aligned} \quad (51)$$

If we introduce the abbreviations

$$x = S_2/S_1, \qquad \delta_1 = a_1 S_1, \qquad \delta_2 = a_2 S_2 \tag{52}$$

equation (51) becomes

$$z^3 + Az^2 + Bz + C = 0 \tag{53}$$

where

$$\left.\begin{aligned} A &= 2\left(x + \frac{1}{x}\right) - \frac{\delta_1\delta_2}{\pi^2} \\ B &= \left(x - \frac{1}{x}\right)^2 + 4\tau^2 + \frac{\delta_1^2 + \delta_2^2}{\pi^2} - \frac{\delta_1\delta_2}{\pi^2}\left(x + \frac{1}{x}\right) \\ C &= \frac{\tau^2}{\pi^2}\left(\frac{\delta_1^2}{x} + \delta_2^2 x - 2\delta_1\delta_2\right) \end{aligned}\right\} \tag{54}$$

Equation (53), in which A, B, and C have the values given in (54), gives the value of z in terms of known constants of the circuits. In these equations, x, δ_1 and δ_2 have the values given in (52).

It is to be borne in mind in using these equations that if a_1 and a_2 are independent of S_1 and S_2 then δ_1 and δ_2 are dependent on S_1 and S_2 and may be dependent on x.

109. If the Original Circuits Are Oscillatory when Each is Alone, All the Real Roots of (53) Are Negative.—As a step toward fixing the limits of z, we shall show that all the real roots of (53) are **negative** provided each of the two original circuits is oscillatory when it is alone and uninfluenced by the other circuit.

If the original circuits are both oscillatory,

$$\delta_1/2\pi < 1, \quad \text{and} \quad \delta_2/2\pi < 1 \tag{55}$$

To prove that the real roots of (53) are negative it is only necessary to show that the coefficients A, B, and C are all positive.

Since $x + 1/x$, where x is positive cannot be less than 2, it is seen that condition (55) makes A positive.

It is seen also that always C is positive, since it is a perfect square.

The remaining coefficient B is more difficult to treat, but may also be shown to be positive under the limitations (55) as follows:

Taking B from (54) add and subtract $2\delta_1\delta_2/\pi^2$, obtaining

$$B = \left(x - \frac{1}{x}\right)^2 + 4\tau^2 + \frac{\delta_1^2 + \delta_2^2}{\pi^2} - \frac{2\delta_1\delta_2}{\pi^2} - \frac{\delta_1\delta_2}{\pi^2}\left(x + \frac{1}{x} - 2\right).$$

The last parenthetical expression may be written in the form

$(x-1)^2/\tau$. Grouping this last term with the first and grouping the fourth term with the third, we obtain

$$B = \frac{(x^2-1)^2}{x^2}\left\{1 - \frac{\delta_1\delta_2}{\pi^2}\frac{x}{(x+1)^2}\right\} + 4\tau^2 + \frac{(\delta_1-\delta_2)^2}{\pi^2} \quad (56)$$

The only term or factor in this equation that is doubtful as to sign is the expression within the brace; but by (55)

$$\frac{\delta_1\delta_2}{\pi^2} < 4 \quad (57)$$

and, since x is positive, it is also apparent that

$$\frac{x}{(x+1)^2} < 1/4 \quad (58)$$

By taking the product of (57) and (58), it is seen that the expression within the brace in the equation (56) for B is positive, and hence B is positive.

We have thus proved that, if each of the original circuits is oscillatory when standing alone, all of the coefficients of the cubic equation (53) *are positive, and that in consequence all of the real values of z are negative.*

We shall next be able to assign certain limits to the value of z, that will simplify the calculation of this quantity.

110. Determination of the Limits of the Value of z for Coupled Circuits Oscillatory when Alone.—In the preceding section we have shown that z is negative provided the original circuits are oscillatory when alone. We can now establish outside limits of the value of z by very simple operations.

To begin, let us take the original definition of z, equation (38), multiply both sides of that equation by S_1S_2, and partly replace $S_1^2S_2^2$ by its value from (37), obtaining

$$S_1S_2z = \frac{a_1a_2S_1^2S_2^2 - a'a''S'^2S''^2}{\pi^2} \quad (59)$$

Let us now make use of the algebraic generalization that for any two real quantities x and y

$$xy \gtreqless \frac{(x+y)^2}{4},$$

then by this relation alone

$$a'a''S'^2S''^2 \gtreqless \frac{(a'S'^2 + a''S''^2)^2}{4} \quad (60)$$

Replacing the right-hand side of (60) by its value from (36) we have

$$a'a''S'^2S''^2 \gtreqless \frac{(a_1S_1^2 + a_2S_2^2)^2}{4} \tag{61}$$

This quantity substituted into (59) gives

$$S_1S_2z \gtreqless -\frac{(a_1S_1^2 - a_2S_2^2)^2}{4\pi^2} \tag{62}$$

Let us recall that z is negative, and let us divide both sides of (62) by S_1S_2, and make use of the abbreviations

$$x = S_2/S_1, \quad \delta_1 = a_1S_1, \quad \delta_2 = a_2S_2, \tag{63}$$

then we obtain

$$0 \gtreqless z \gtreqless -\frac{\left(\frac{\delta_1}{\sqrt{x}} - \delta_2\sqrt{x}\right)^2}{4\pi^2} \tag{64}$$

The inequality (64) *gives the limits of the value of* z, *provided the original circuits are oscillatory when not coupled.*

111. Reduction of the Cubic Equation for z to a Quadratic Equation over an Important Range of Constants.—In equation (53) we have given a cubic equation for the determination of z, and we have shown that z is negative, and that it has the limiting values specified by the inequality (64), provided the original circuits are oscillatory, that is, provided

$$\frac{\delta_1\delta_2}{4\pi^2} < 1 \tag{65}$$

In the cubic equation

$$z^3 + Az^2 + Bz + C = 0$$

the terms z^3 and Bz are the only negative terms. It thus appears that we can neglect z^3 provided it is negligible in comparison with the other negative term Bz; that is, provided

$$z^2 << B \tag{66}$$

and this proposition can be tested by making use of the limiting value of z from (64) and comparing this limiting value when squared with B from (54).

It is to be noted that when x is unity the maximum possible value of z^2 is of the order of $(\delta_1 - \delta_2)^4/16\pi^4$, while the order of B is $4\tau^2 + (\delta_1 - \delta_2)^2/\pi^2$, so that z^2 is negligible in comparison with B, provided

$$\frac{(\delta_1 - \delta_2)^2}{16\pi^2} << 1 \tag{67}$$

We thus see in a general way that z^2 is likely to be negligible in comparison with B. A careful examination of the possible values of z^2 and B over the whole possible range of constants of the circuits, shows that occasions may arise in which (66) is not fulfilled, so that we then require the whole cubic to determine z.

In cases, on the other hand, in which z^2 is negligible in comparison with B, the cubic reduces to the quadratic

$$Az^2 + Bz + C = 0 \tag{68}$$

of which the solution is

$$z = -\frac{B}{2A}\left\{1 - \sqrt{1 - \frac{4AC}{B^2}}\right\} \tag{69}$$

In this solution we have chosen the minus sign before the radical, because this gives the smaller absolute value of z as is required by the condition that z^3 be negligible. The question of this sign is investigated in certain of the special cases treated below, but has not been given any extended general investigation.

We may sum up regarding z as follows: z is exactly given by the cubic equation (53). *Whenever z is so small that its square is negligible in comparison with the coefficient B, as is often the case, the value of z is given with sufficient accuracy by* (69). *Even if the whole cubic must be used in determining z, the calculations may be facilitated by making a preliminary approximate calculation of z by* (69).

We have now solved completely the problem of determining the damping constants and periods of the coupled system.

We shall now proceed to a numerical treatment of certain important special cases, and as a result of the calculations we shall have our attention called to important simplifications that sometimes arise.

The special cases to be investigated are as follows:

Case I. The Quasi Isochronous System,
Case II. The General Case with Numerical Constants, and
Case III. The Loose-coupled System.

CASE 1. THE QUASI ISOCHRONOUS SYSTEM

112. The Equations for z in the Quasi Isochronous System. We shall now limit the discussion to the case in which the original two circuits have nearly the same free periods T_1 and T_2. In-

stead of assuming T_1 exactly equal to T_2, it is simpler to assume the undamped periods equal; that is

$$S_1 = S_2 = S \text{ (say)} \tag{70}$$

We shall call this the case of **quasi isochronism.**

To avoid continually writing certain combinations of δ_1 and δ_2, we shall use the following abbreviations,

$$u = \frac{(\delta_1 - \delta_2)^2}{4\pi^2}, \qquad v = \frac{\delta_1\delta_2}{4\pi^2} \tag{71}$$

Under the condition of quasi isochronism the quantities defined in (54) become

$$x = 1, \; A = 4(1 - v), \; B = 4(\tau^2 + u), \; C = 4\tau^2 u \tag{72}$$

The cubic equation (53) for z, in the isochronous case, becomes

$$z^3 + 4z^2(1 - v) + 4z(\tau^2 + u) + 4\tau^2 u = 0 \tag{73}$$

This factors into

$$\frac{z^2}{4}(z - 4v) + (z + u)(\tau^2 + z) = 0 \tag{74}$$

From this factored form we can make a discovery of a new fact in regard to the limit of z. We have already shown in the general case the relation (64), which in the isochronous case (since $x = 1$) becomes

$$0 \geqq z \geqq -u \tag{75}$$

This inequality may be otherwise written in the form

$$z \leqq 0, \quad \text{and } z + u \geqq 0 \tag{76}$$

This fact applied to (74) shows that the first term of that equation is negative or zero, and therefore the last term must be positive or zero to make the sum zero. We have just shown in (76) that one of the factors $(z + u)$ of the last term is positive or zero, and hence the other factor of that term is positive or zero; that is,

$$z + \tau^2 \geqq 0 \tag{77}$$

This result is important in determining the signs and the limiting values of expressions to follow.

We shall now examine the condition under which the cubic

equation for z reduces to a quadratic. This condition as stated in (66) may be written

$$z^2 << B,$$

which in the isochronous case, by (72), becomes

$$z^2 << 4(\tau^2 + u) \tag{78}$$

Now by (75) and (77),

$$z^2 < u^2 \text{ and also } z^2 < \tau^4$$

so that (78) is met if

$$\text{either } u^2 << 4(\tau^2 + u) \text{ or } \tau^4 << 4(\tau^2 + u) \tag{79}$$

Either of the alternative conditions of (79) is sufficient to reduce the cubic to the quadratic. If $u < \tau^2$ the first of the alternatives is met if

$$u^2 << 8u; \text{ that is } \frac{(\delta_1 - \delta_2)^2}{32\pi^2} << 1 \tag{80}$$

If $\tau^2 < u$, the second alternative of (79) is met if

$$\frac{\tau^2}{8} << 1 \tag{81}$$

We have then the result that the quadratic relation is sufficient to determine z in any case of quasi isochronism in which $\tau^2/8$ is negligible in comparison with unity or in which $(\delta_1 - \delta_2)^2/32\pi^2$ is negligible in comparison with unity. The latter of these alternatives is true for oscillatory circuits even when they are very highly damped. The exact degree of damping is easily determined in a specific case.

Let us now write out the simplified value of z for the isochronous system. This is done by replacing the coefficients A, B, and C in (69) by their values from (72), and gives

$$z = -\frac{\tau^2 + u}{2(1 - v)}\left\{1 - \sqrt{1 - \frac{4\tau^2 u(1 - v)}{(\tau^2 + u)^2}}\right\}$$

$$= -\frac{\tau^2 + u}{2(1 - v)} + \sqrt{\frac{(\tau^2 - u)^2 + 4\tau^2 uv}{4(1 - v)^2}}. \tag{82}$$

Let us now make a digression to prove the correctness of the signs before the radicals in (82), since this matter was passed

over without much attention in the statements following (69). Using the second of the forms of (82), transposing the first term of the right-hand side to the left-hand side, and collecting terms over a common denominator, we have

$$\frac{2z(1 - v) + \tau^2 + u}{2(1 - v)} = R,$$

where R is a temporary abbreviation for the radical. The numerator of the left-hand side can be regrouped giving

$$\frac{(z + \tau^2) + (z + u) - 2zv}{2(1 - v)} = R.$$

Now by (76) and (77) it is seen that all the terms of the numerator are positive, so that the radical R must be positive (in the second form of (82)), provided v is less than unity; that is, provided the original circuits are oscillatory. Hence the correctness of the signs given to the radicals in (82).

In the case in which $S_1 = S_2 = S$ the quantity z is exactly given by (73), *or* (74). *In all cases in which the condition* (80), *or the condition* (81), *is fulfilled z is given with sufficient accuracy by* (82). *For these isochronous circuits,* (82) *is of almost universal applicability.*

It may be noted also, if the original circuits are separately oscillatory, that the absolute value of z is less than u and less than τ^2 and that z is negative, as is shown by (76) *and* (77).

113. The Equations for Undamped Periods and Damping Constants in the Quasi Isochronous Circuits.—With the original circuits separately tuned to the same undamped periods so that

$$S_1 = S_2 = S,$$

the equations (41) and (42) for the undamped periods of the coupled system, on being squared, give

$$\left.\begin{aligned} S'^2 &= S^2\left\{1 + \frac{z}{2} + \sqrt{\tau^2 + z + \frac{z^2}{4}}\right\} \\ S''^2 &= S^2\left\{1 + \frac{z}{2} - \sqrt{\tau^2 + z + \frac{z^2}{4}}\right\} \end{aligned}\right\} \tag{83}$$

For the damping constants we shall use the equations (48) and (49), but we shall first examine the criterion (47) as to the

sign to use before the radicals in (48) and (49). In the case of isochronism, the criterion (47) reduces to

$$\frac{(a_1 + a_2)(2 + z)}{2(1 - \tau^2)} > a_1 + a_2,$$

which reduces to

$$z + 2\tau^2 > 0.$$

By (77) this condition is always fulfilled with oscillatory circuits, and, therefore, by the note following (48) and (49) the signs in these two equations are correct for this case.

These two equations (48) and (49), by the isochronous condition $S_1 = S_2 = S$, reduce to

$$a' = \frac{a_1 + a_2}{2(1 - \tau^2)} \{1 - \sqrt{1 - \theta}\} \tag{84}$$

$$a'' = \frac{a_1 + a_2}{2(1 - \tau^2)} \{1 + \sqrt{1 - \theta}\} \tag{85}$$

where

$$\theta = \frac{(1 - \tau^2)(4v - z)}{4v + u} \tag{86}$$

In the system of two circuits that are separately tuned to the same undamped periods, the resultant undamped periods when the circuits are coupled together and allowed to oscillate freely are given by the equations (83), *and the resultant damping constants are given by* (84) *and* (85) *in terms of* θ *defined by* (86). *The values of* u *and* v *are defined in* (71). z *is given by* (73) *and is usually given with sufficient accuracy by* (82).

114. Application to Two Numerical Cases of Quasi Isochronism. As the first special case of quasi isochronism, let us take the following numerical values.

Let

$$\delta_1 = 0.3\pi, \qquad \delta_2 = 0.1\pi \tag{87}$$

In this case the values of u and v become

$$u = 0.01, \quad v = 0.0075, \quad 4v + u = 0.04 \tag{88}$$

With these numerical values (82) becomes

$$z = -\frac{\tau^2 + 0.01}{1.985}\left\{1 - \sqrt{1 - \frac{0\ 0397\tau^2}{(\tau^2 + 0.01)^2}}\right\}.$$

In this numerical case (84) and (85), on multiplying both sides by S/π, become

$$\frac{a'S}{\pi} = \frac{0.2}{1-\tau^2}\{1 - \sqrt{1-\theta}\} \tag{89}$$

and

$$\frac{a''S}{\pi} = \frac{0.2}{1-\tau^2}\{1 + \sqrt{1-\theta}\} \tag{90}$$

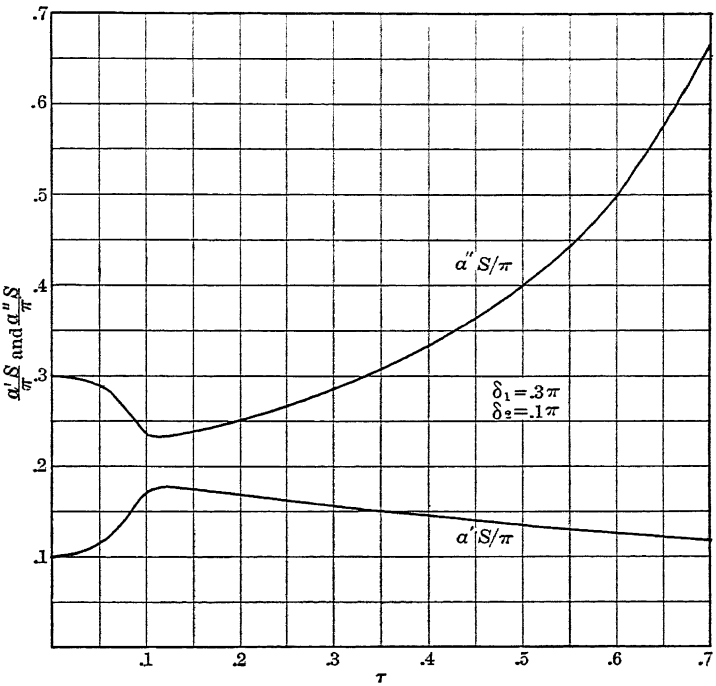

FIG. 2.—Quantities proportional to resultant damping constants plotted against coefficient of coupling τ in special case in which $\delta_1 = 0.3\pi$, $\delta_2 = 0.1\pi$, and $S_1 = S_2 = S$.

where

$$\theta = \frac{(1-\tau^2)(0.03 - z)}{0.04} \tag{91}$$

Computations were made for various values of τ. The method of making the computations consists in first determining z by the use of the equation following (88) and then computing $a'S/\pi$ and $a''S/\pi$ by the use of (89), (90) and (91). The values of S'/S and S''/S may be computed directly from (83).

Table I.—Computed Values of Damping Constants and Undamped Periods of the Quasi Isochronous System of Two Circuits with Various Values of τ. Given $\delta_1 = 0.3\pi$, $\delta_2 = 0.1\pi$

τ	τ^2	z	$\frac{a''S}{\pi}$	$\frac{a'S}{\pi}$	$\frac{S'}{S}$	$\frac{S''}{S}$	$\frac{S'}{S\sqrt{1+\tau}}$	$\frac{S''}{S\sqrt{1-\tau}}$
0 0000	0 0000	−0 00000	0 300	0 100	1 0000	1 0000	1 000	1 000
0 0278	0 00077	−0 00077	0 296	0 104	1 0002	0 9994	0 986	1 011
0 0578	0 00334	−0 00333	0 283	0 118	1 0011	0 9962	0 973	1 025
0 0802	0 00646	−0 00636	0 264	0 139	1 0037	0 9931	0 965	1 034
0 092	0 00842	−0 00810	0 249	0 154	1 0071	0 9887	0 957	1 036
0 101	0 0103	−0 00922	0 237	0 167	1 0143	0 9809	0 965	1 033
0 109	0 0119	−0 00961	0 232	0 173	1 0216	0 9731	0 970	1 029
0 121	0 0146	−0 00980	0 232	0 175	1 0323	0 9621	0 976	1 025
0 187	0 0348	−0 00996	0 246	0 168	1 0735	0 9151	0 984	1 014
0 200	0 0400	−0 01000	0 250	0 167	1 0810	0 9065	0 986	1 012
0 300	0 0900	−0 01000	0 286	0 154	1 130	0 8439	0 990	1 007
0 400	0 1600	−0 01000	0 333	0 143	1 176	0 7796	0 993	1 005
0 500	0 2500	−0 01000	0 400	0 133	1 219	0 7107	0 995	1 003
0 600	0 3600	−0 01000	0 500	0 125	1 260	0 6353	0 996	1 002
0 700	0 4900	−0 01000	0 667	0 118	1 299	0 5497	0 996	1 002
0 800	0 6400	−0 01000	1 000	0 113	1 337	0 4487	0 997	1 002
0 900	0 8100	−0 01000	2 000	0 105	1 374	0 3172	0 998	1 001
1 000	1 000	−0 01000	infin	0 100	1 414	0 0000	1 000	1 000

Table II.—Computed Values of Damping Constants and Undamped Periods of the Quasi Isochronous System of Circuits with Various Values of τ. Given $\delta_1 = 0.03\pi$, $\delta_2 = 0.01\pi$

τ	τ^2	z	$\frac{a'S}{\pi}$	$\frac{a''S}{\pi}$	$\frac{S'}{S}$	$\frac{S''}{S}$	$\frac{S'}{S\sqrt{1+\tau}}$	$\frac{S''}{S\sqrt{1-\tau}}$
0 000	0 000000	−0 000000000	0 01000	0 03000	1 0000	1 0000	1 0000	1 0000
0 001	0 000001	−0 000001000	0 01005	0 02995	1 0000	1 0000	0 9995	1 0005
0 002	0 000004	−0 000003999	0 01020	0 02980	1 0000	1 0000	0 9990	1 0010
0 004	0 000016	−0 000015996	0 01083	0 02917	1 0000	0 9999	0 9980	1 0019
0 006	0 000036	−0 00003598	0 01200	0 02800	1 0001	0 9999	0 9970	1 0029
0 008	0 000064	−0 00006394	0 01390	0 02601	1 0001	0 9999	0 9960	1 0039
0 010	0 000100	−0 00009974	0 01940	0 02057	1 0002	0 9997	0 9953	1 0047
0 012	0 000144	−0 00009997	0 01996	0 02024	1 0031	0 9964	0 9970	1 0026
0 015	0 000225	−0 00009999	0 01970	0 02030	1 0053	0 9941	0 9978	1 0017
0 020	0 000400	−0 00010000	0 01960	0 02041	1 0084	0 9911	0 9984	1 0011
0 030	0 000900	−0 00010000	0 01941	0 02061	1 014	0 9855	0 9989	1 0006
0 040	0 001600	−0 00010000	0 01923	0 02083	1 019	0 9803	0 9991	1 0004
0 050	0 002500	−0 00010000	1 01905	0 02105	1 024	0 9750	0 9993	1 0002
0 1	0 01	−0 00010000	0 01818	0 02222	1 048	0 9487	0 9995	1 0000
0 2	0 04	−0 00010000	0 01666	0 02500	1 095	0 8943	0 9997	0 9999
0 3	0 09	−0 00010000	0 01538	0 02856	1 140	0 8365	0 9997	0 9998
0 4	0 16	−0 00010000	0 01437	0 03333	1 183	0 7744	0 9998	0 9997
0 5	0 25	−0 00010000	0 01333	0 04000	1 224	0 7088	0 9998	0 9996
0 6	0 36	−0 00010000	0 01250	0 05000	1 264	0 6318	0 9998	0 9994
0 7	0 49	−0 00010000	0 01176	0 06667	1 303	0 5474	0 9998	0 9993
0 8	0 64	−0 00010000	0 01111	0 10000	1 341	0 4468	0 9998	0 9989
0 9	0 81	−0 00010000	0 01052	0 20000	1 378	0 3155	0 9999	0 9978
1 0	1 00	−0 00010000	0 01000	infin	1 414	0 0000	1 0000	1 0000

Table I contains the results of the calculation with various values of τ. These results are plotted in the curves of Figs. 2 and 3.

As a second example of the quasi isochronous system, we have computed the case in which

$$\delta_1 = 0.03\pi, \qquad \delta_2 = 0.01\pi \tag{92}$$

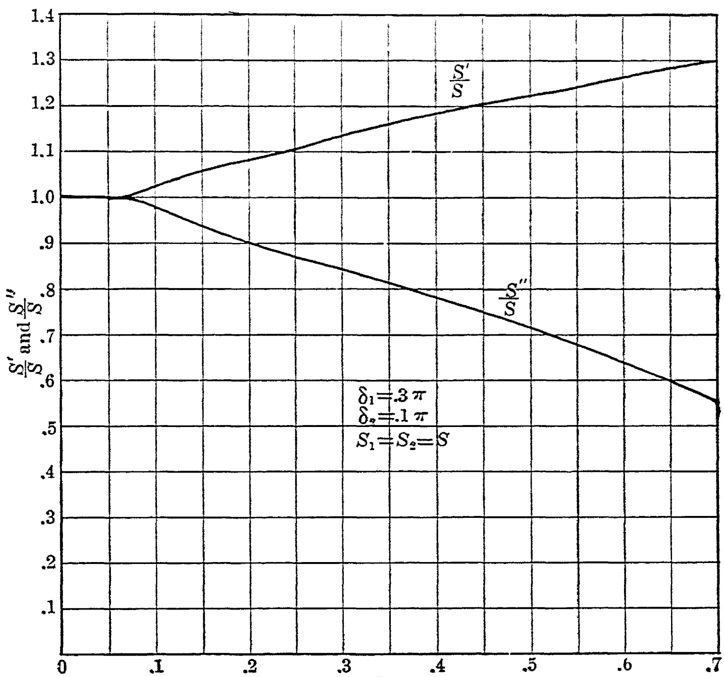

FIG. 3.—Quantities proportional to S' and S'' plotted against τ in special case in which $\delta_1 = 0.3\pi$, $\delta_2 = 0.1\pi$, and $S_1 = S_2 = S$

The results in this case are recorded in Table II and some of the significant values are plotted in Figs. 4 and 5. Although the scale in Figs. 4 and 5 is different from the scale in Figs. 2 and 3, it is seen that the case with the decrements given in (92) has general characteristics in common with the case with the larger decrements given in equation (87).

115. Discussion of the Results in the Numerical Cases of Isochronous Circuits, with Derivation of Limiting Values of z. Certain significant facts are apparent from Tables I and II, compiled for the two sets of specific values of the decrements.

One of these facts is that for small values of τ^2, z is approximately equal to $-\tau^2$. This may be derived theoretically from the cubic equation (74) for z, which by transposition of the first term to the right and division by $z + u$ gives

$$z = -\tau^2 + \frac{z^2(4v - z)}{4(z + u)}.$$

If z is to become approximately $-\tau^2$ the second term of the right-hand side must be small, and the equation must still be

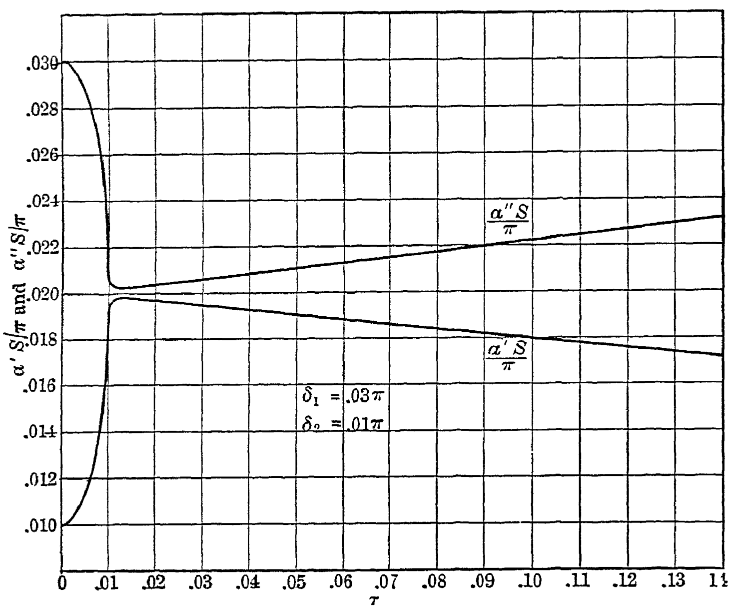

FIG. 4.—Same as Fig 2 except that $\delta_1 = 0.03\pi$, $\delta_2 = 0.01\pi$, and that scale is changed.

approximately correct when z in the fraction is replaced by $-\tau^2$, giving

$$z = -\tau^2\{1 - \frac{\tau^2(4v + \tau^2)}{4(u - \tau^2)}\}, \text{ approximately.} \tag{93}$$

Now by (77) $z + \tau^2$ must be positive, so that (93) can be employed only when τ^2 is less than u, and since the fraction of (93) was obtained by replacing z by $-\tau^2$, it is seen that for (93) to be applicable the fraction in (93) must be small in comparison with unity. If these conditions are fulfilled z becomes approximately

equal to $-\tau^2$. In symbols, these statements may be written as follows:

$$\text{If } \tau^2 < u, \text{ and if } \frac{\tau^2(4v + \tau^2)}{4(u - \tau^2)} << 1 \tag{94}$$

then

$$z = -\tau^2 \tag{95}$$

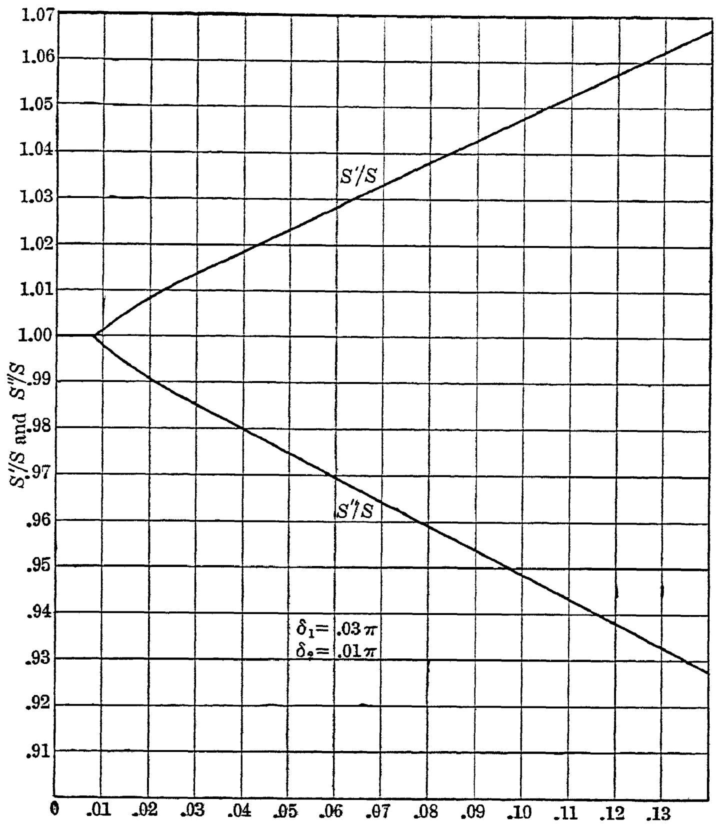

FIG. 5—Same as Fig. 3, except that $\delta_1 = 0.03\pi$, $\delta_2 = 0.01\pi$, and scale is changed

With the quasi isochronous system of circuits, and under the conditions expressed in (94) *z may be equated to* $-\tau^2$ *as given in* (95). *In subsequent sections we shall designate the case in which* (94) *and* (95) *are fulfilled as the* τ-**case.**

Another fact apparent from Tables I and II is that with increasing values of τ, z approaches in each case a definite limit, and this definite limit in each case is seen to be $-u$.

This result may also be established analytically, as follows:

Transposing the first term of (74) to the right, and dividing the resulting equation by $\tau^2 + z$, we obtain

$$z = -u + \frac{z^2(4v - z)}{4(\tau^2 + z)},$$

which reduces to u provided the last term is negligible and positive. In detail, let us replace z by $-u$ in the last term, obtaining

$$z = -u\left\{1 - \frac{u(4v + u)}{4(\tau^2 - u)}\right\}, \text{ approximately.} \tag{96}$$

But, since $z + u$ is positive by (76), this can only be true provided u is less than τ^2, and since we have replaced z by $-u$, we must require also that the last term in (96) be negligible. In symbols, we have

$$\text{If} \qquad u < \tau^2, \text{ and if } \frac{u(4v + u)}{4(\tau^2 - u)} << 1 \tag{97}$$

$$\text{then} \qquad z = -u = -\frac{(\delta_1 - \delta_2)^2}{4\pi^2} \tag{98}$$

The last step of (98) is by the definition of u given in (71)

With the quasi isochronous system of circuits, and under the conditions expressed in (97), *z may be equated to $-u$. In subsequent sections we shall designate this case as the* **u-case.**

116. Simplified Equations for the Damping Constants and Periods in the u-Case and the τ-Case of Isochronous Circuits. In the u-case and the τ-case as described in the preceding section, z reduces to very simple values, and the damping constants and undamped periods may be also expressed in simplified form. We shall take the two cases in order beginning with the u-case.

117. The u-Case.—Equations (97) and (98) give the relations of the constants in the u-case. Replacing u and v by their values from (71) and (72), we obtain

If

$$\frac{(\delta_1 - \delta_2)^2}{4\pi^2} < \tau^2, \text{ and if } \frac{(\delta_1^2 - \delta_2^2)^2}{64\pi^4} << \tau^2 - \frac{(\delta_1 - \delta_2)^2}{4\pi^2} \tag{99}$$

then

$$z = -\frac{(\delta_1 - \delta_2)^2}{4\pi^2} \tag{100}$$

Introducing the value $z = -u$ into (86) we obtain

$$\theta = 1 - \tau^2,$$

whence (84) and (85) become

$$a' = \frac{a_1 + a_2}{2(1 + \tau)}, \text{ and } a'' = \frac{a_1 + a_2}{2(1 - \tau)} \tag{101}$$

Under the conditions set forth in (99), *or in abbreviated form in* (97), *equation* (101) *gives the damping constants in the isochronous system of two magnetically coupled circuits This we have called the u-case.*

In this u-case, the undamped period equations (83) become by (98)

$$S'^2 = S^2\left\{1 - \frac{u}{2} + \sqrt{\tau^2 - u + \frac{u^2}{4}}\right\} \tag{102}$$

$$S''^2 = S^2\left\{1 - \frac{u}{2} - \sqrt{\tau^2 - u + \frac{u^2}{4}}\right\} \tag{103}$$

In the u-case of isochronous circuits, as specified by (99), or (97), *equations* (102) *and* (103) *give the squares of the undamped periods in the coupled system. The value of u is given in* (71).

Note that as u approaches zero in this case, S' and S'' approach the values

$$S' = S\sqrt{1 + \tau}, \text{ and } S'' = S\sqrt{1 - \tau} \tag{104}$$

and in the special case in which δ_1 and δ_2 are made zero, S, S' and S'' are respectively equal to T, T' and T'', so that we have

$$T' = T\sqrt{1 + \tau}, \text{ and } T'' = T\sqrt{1 - \tau} \tag{105}$$

as is required by Chapter VII.

Equation (105) *gives the values of the periods in the isochronous system in which δ_1 and δ_2 are zero, and is in agreement with Chapter VII. Equation* (104) *gives the undamped periods in the quasi isochronous circuits when $\delta_1 = \delta_2$.*

118. The τ-Case.—This designation applies to the case in which $z = -\tau^2$. The conditions for this are given in (94). On replacing u and v by their values from (71), (94) and (95) become

If

$$\tau^2 < \frac{(\delta_1 - \delta_2)^2}{4\pi^2}, \text{ and if } \tau^2\left\{\frac{\delta_1\delta_2}{4\pi^2} + \frac{\tau^2}{4}\right\} << \frac{(\delta_1 - \delta_2)^2}{4\pi^2} - \tau^2 \tag{106}$$

then

$$z = -\tau^2 \tag{107}$$

To obtain the damping constants in this case, let us substitute (107) into (86), obtaining

$$\theta = \frac{(1 - \tau^2)(4v + \tau^2)}{4v + u},$$

which substituted into (84) and (85) gives

$$a' = \frac{a_1 + a_2}{2(1 - \tau^2)}\left\{1 - \sqrt{\frac{u - \tau^2 + \tau^4 + 4v\tau^2}{4v + u}}\right\}$$

$$= \frac{a_1 + a_2}{2(1 - \tau^2)}\left\{1 - \sqrt{\frac{u - \tau^2}{4v + u}}\right\}, \text{ approximately,} \qquad (108)$$

$$a'' = \frac{a_1 + a_2}{2(1 - \tau^2)}\left\{1 + \sqrt{\frac{u - \tau^2 + \tau^4 + 4\tau^2}{4v + u}}\right\}$$

$$= \frac{a_1 + a_2}{2(1 - \tau^2)}\left\{1 + \sqrt{\frac{u - \tau^2}{4v + u}}\right\}, \text{ approximately} \qquad (109)$$

In the τ-case of isochronous circuits, as specified in (106), *equations* (108) *and* (109) *give the damping constants in the coupled system. In the values marked "approximately" we have neglected a quantity twice as large as that specified as negligible in* (106). *The values of u and v are given in* (71).

Taking up next the undamped periods in this τ-case and replacing z by $-\tau^2$ in (83), we obtain

$$S'^2 = S^2, \quad \text{and } S''^2 = S^2(1 - \tau^2) \qquad (110)$$

These results may be inaccurate, since in (83) the radical involves the sum of z and τ^2 and also involves z^2. We can obtain a closer approximation by employing for z equation (93), giving

$$z + \tau^2 = \frac{\tau^2(4v + \tau^2)}{4(u - \tau^2)}$$

Now adding to this $z^2/4 = \tau^4/4$, approximately, we have

$$z + \tau^2 + z^2/4 = \frac{\tau^4(4v + u)}{4(u - \tau^2)}.$$

This inserted into (83) gives

$$S'^2 = S^2\left\{1 - \frac{\tau^2}{2} + \frac{\tau^2}{2}\sqrt{\frac{4v + u}{u - \tau^2}}\right\} \qquad (111)$$

$$S''^2 = S^2\left\{1 - \frac{\tau^2}{2} - \frac{\tau^2}{2}\sqrt{\frac{4v + u}{u - \tau^2}}\right\} \qquad (112)$$

Equations (111) *and* (112) *give the values of the undamped periods (squared) for the two isochronous circuits in the τ-case,*

as specified in (94), *or* (106). *The values of u and v are given in* (71).

119. τ-Case, Continued. Limits Approached as τ^2 Approaches Zero.—In the preceding section we have given equations for the damping constants and undamped periods in what has been called the τ-case, as specified by (94), or (106) Let us now suppose that τ^2 is small enough to be neglected in (108), (109), (111), and (112); then these equations reduce to

$$a' = a_2, \quad a'' = a_1, \quad S' = S'' = S \tag{113}$$

as may be seen by making $\tau^2 = 0$, and replacing u and v by their values

The condition under which τ^2 is sufficiently near zero to make (113) substantially correct, may be derived by examining (108). Expansion of the radical in (108), second form, gives

$$\begin{aligned}\sqrt{\frac{u - \tau^2}{4v + u}} &= \sqrt{\frac{u}{4v + u}}\left\{1 - \frac{\tau^2(4v + u)}{2(u)} - \cdots\right\} \\ &= \frac{a_1 - a_2}{a_1 + a_2}\left\{1 - \frac{\tau^2(4v + u)}{2u} - \cdots\right\} \\ &= \frac{a_1 - a_2}{a_1 + a_2}\end{aligned} \tag{114}$$

provided

$$\tau^2 << \frac{2u}{4v + u} = \frac{2(a_1 - a_2)^2}{(a_1 + a_2)^2} \tag{115}$$

In making these reductions we have used the definitions of u and v given in (71), and have used also the definitions (52) of δ_1 and δ_2 with $S_1 = S_2 = S$ for this special case of isochronous circuits.

The remaining step of reducing a' to a_2 and a'' to a_1, as given in (113), consists in substituting (114) into (108), and making τ^2 negligible in comparison with 1.

Equations (113) *give the damping constant and undamped periods in the isochronous system, provided τ^2 is negligible as specified in* (115).

120. Summary of Results with the Quasi Isochronous System of Two Magnetically Coupled Circuits.—Considering first the damping constants, and having reference to Figs. 2 and 4, it is seen that for small values of τ^2, as specified in (115),

$$a' = a_2, \quad a'' = a_1.$$

Under this same condition of small τ^2, with however, a somewhat larger possible value of τ^2, reference to the Tables I and II, and to the curves of Figs 3 and 5, and to the analysis of the preceding section, shows that substantially

$$S' = S'' = S.$$

For larger values of τ^2, such as are specified in (99) and designated the u-case, a' and a'' are given by (101); namely

$$a' = \frac{a_1 + a_2}{2(1 + \tau)} \text{ and } a'' = \frac{a_1 + a_2}{2(1 - \tau)}$$

Referring to Tables I and II, and to Figs. 2 and 4, it is seen that this latter condition is attained for values of τ greater than about twice the values of τ at which the a' curve and the a'' curve come nearest together to form a neck in the figures.

For this same range of values of τ, in which τ is greater than twice the value at which the neck is formed by the a' and a'' curves, S' and S'' are given by (102) and (103), and in the special case of small values of u (that is, small values of $(\delta_1 - \delta_2)^2/4\pi^2$) these quantities are approximately given by (104), which is

$$S' = S\sqrt{1 + \tau}, \text{ and } S'' = S\sqrt{1 - \tau}.$$

For values of τ intermediate between those values that give the simplified expressions for damping constants and periods, the exact expressions involving z must be employed.

CASE II. THE GENERAL CASE WITH NUMERICAL COEFFICIENTS

121. Statement.—If we take the general case of two magnetically coupled circuits, such as are shown in Fig. 1, and suppose that the two separate circuits, when each is standing alone have the undamped periods S_1 and S_2 and the damping constants a_1 and a_2, the equations (41) and (42) specify the values of the undamped periods that coexist in both of the circuits when they are coupled together with a coefficient of coupling τ. The equations (45) and (46) give the damping factors in the two oscillations of the coupled system.

Both of these pairs of equations involve a quantity z. The exact value of z is given by the cubic equation (53) which has coefficients A, B, and C defined in (54). If we know the coefficient of coupling τ, the decrements δ_1 and δ_2 of the original circuits, and x, which is the ratio of S_2 to S_1, we can compute z

from (53), and can then proceed to solve completely the problem of finding the periods and damping factors of the coupled system.

Instead of using the cubic equation (53) for z, it is usually sufficiently accurate to use the values of z given by (69). The test of this point is specified in (66).

We shall now proceed to compute S', S'', a', and a'' for four different values of τ^2, and shall allow the ratio of S_2 to S_1 to be varied by varying S_2, while S_1 is kept constant. With this con-

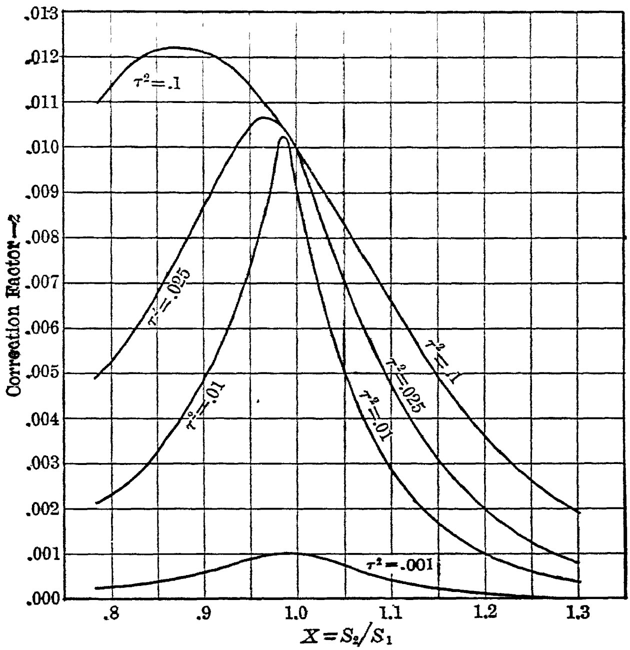

FIG. 6.—Circuits not isochronous. Values of correction factor ($-z$) for various values of S_2/S_1 and for various values of τ.

dition, if a_1 and a_2 are supposed to remain constant, δ_1, which is a_1S_1, will stay constant, but δ_2, which is a_2S_2 will vary. We shall therefore assign a fixed numerical value 0.3π to δ_1, and shall assign a fixed value 0.1π to δ_2 at $S_2 = S_1$. That is $\delta_1 = 0.3\pi$, and $a_2S_1 = 0.1\pi$.

122. Computation of z in the General Case of Two Magnetically Coupled Circuits with Given Values of δ_1, a_2S_1, and with Various Values of τ^2 and Various Values of the Ratio of S_2 to S_1.—We shall take in our numerical illustration

$$\delta_1 = 0.3\pi, \quad a_2S_1 = 0.1\pi, \quad \text{then } \delta_2 = 0.1\pi x \tag{113}$$

where, as in (63)

$$x = S_2\ S_1 \tag{114}$$

The coefficients A, B, and C of (54), in this numerical case, become

$$A = 1.97x + \frac{2}{x} \tag{115}$$

$$B = 4\tau^2 + \frac{1}{x^2} + 0\ 98x^2 - 1.94 \tag{116}$$

$$C = \tau^2\left(0.01x^3 + \frac{0.09}{x} - 0\ 06x\right) \tag{117}$$

The quantity τ^2 is given four values; namely, 0.1, 0.01, 0.025, and 0.001.

The first computation consisted in determining z. For this purpose the reduced equation (82), or (69), has been sufficient for all values of the computation, except for two values that are indicated in the table, where it was found necessary to use the cubic (53) instead of the reduced equation.

The results for z are given in Table III, and are plotted in the curves of Fig. 6.

Table III.—Computed Values of the Correction Factor z in the Special Case in Which $\delta_1 = 0.3\pi$, $a_2S_1 = 0.1\pi$ for the General Case with $S_2/S_1 = x$, and with Four Different Values of τ^2

$x = S_2/S_1$	Values of $-z$ for			
	$\tau^2 = 0\ 001$	$\tau^2 = 0\ 01$	$\tau^2 = 0.025$	$\tau^2 = 0\ 1$
0 76923	0 000231	0 002113	0 004587	0 011006
0.83333	0 0003287	0 002892	0 006250	0 012031
0 90909	0.0006382	0.005264	0 009028	0.012031
0 95238	0.0008865	0.007436	0 010424	0 011313
0 96154	0.0009358	0 008066	0 010619[1]	0 011104
0 97087	0.0009728	0.008687	0 010641	0 010858
0 98039	0 0010013	0.010108	0 010568	0 010592
0.99010	0 0009760	0 010165[1]	0 010314	0 010302
1 00	0 0009991	0 009203	0 009950	0 009991
1 01	0.0009659	0.008009	0 009478	0 009648
1 02	0.0009160	0 007245	0 008945	0.009325
1 03	0.0008499	0 006410	0.008175	0.008978
1 04	0.0007794	0.006053	0.007792	0.008622
1 05	0.0007147	0.005206	0.007214	0.008403
1 10	0.0003955	0.002891	0.004750	0.006529
1 20	0.0001199	0.000997	0.001967	0 003683
1 30	0.0000419	0.000382	0.000816	0 001882

[1] In computing these two values all the terms of the cubic equation (53) were used.

123. Computation of S′ and S″ in the General Case with Numerical Constants.—Having computed the values of $-z$ recorded in Table III, we shall next make numerical computations of S' and S''. For this purpose, we shall divide both sides of (41) and (42) by S_1, and replace S_2/S_1 by x, obtaining

$$\frac{S'}{S_1} = \frac{1}{2}\sqrt{1 + x^2 + xz + 2x\sqrt{1 - \tau^2}} + \frac{1}{2}\sqrt{1 + x^2 + xz - 2x\sqrt{1 - \tau^2}} \qquad (118)$$

$$\frac{S''}{S_1} = \frac{1}{2}\sqrt{1 + x^2 + xz + 2x\sqrt{1 - \tau^2}} - \frac{1}{2}\sqrt{1 + x^2 + xz - 2x\sqrt{1 - \tau^2}} \qquad (119)$$

Using the values of x, $-z$, and τ^2 given in Table III, the values recorded in Tables IV, V, VI and VII in the columns marked S'/S_1 and S''/S_1 were obtained. These values are plotted in Figs. 7 to 10.

For comparison, to show the effect of the damping constants in modifying the periods, there is recorded in parentheses after each value of S'/S_1 and S''/S_1 the value obtained by regarding z as zero. In Fig. 8 the dotted curve is a graph of values obtained by neglecting z, while the continuous line curve is the graph of true values with z considered.

124. Computation of a′ and a″ in the General Case with Numerical Constants.—Continuing with the same set of special values, we have next computed the values of ratios expressing a' and a'' in terms of known quantities.

For the formulation of this problem, let us first examine the equation (47), which is used to determine the algebraic signs of certain damping constant equations to be employed. Dividing both sides of the inequality (47) by S_1, and replacing S_2/S_1 by x, we obtain

$$\frac{(a_1S_1 + a_2S_1)(1 + x^2 + xz)}{2(1 - \tau^2)} > a_1S_1 + a_2S_1x^2 \qquad (120)$$

Replacing a_1S_1 by its special value 0.3π, and a_2S_1 by its special value 0.1π, we obtain

$$0.2(1 + x^2 + xz) > (1 - \tau^2)(0.3 + 0.1x^2) \qquad (121)$$

as the criterion for determining the signs in (48) and (49), which we are going to employ. If (121) is fulfilled, the signs in (48)

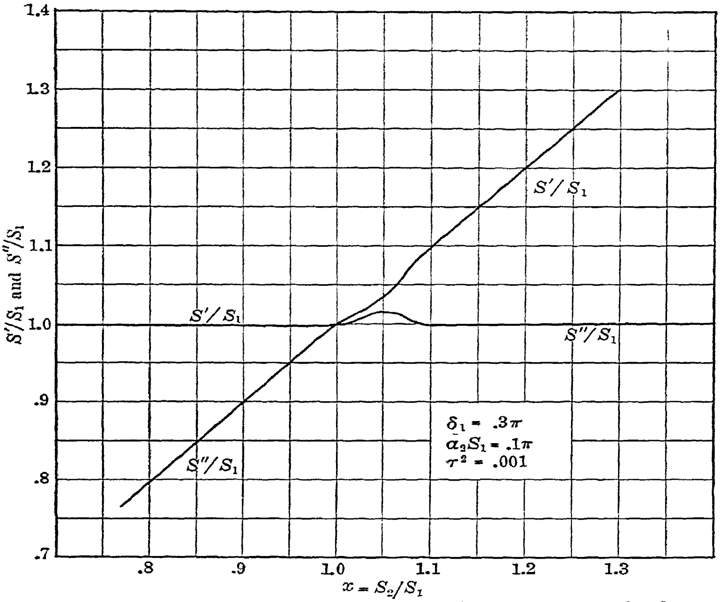

FIG 7.—Curves of ratios of undamped periods for circuits not isochronous, with $\delta_1 = 0.3\pi$, $a_2S_1 = 0.1\pi$, $\tau^2 = 0.001$.

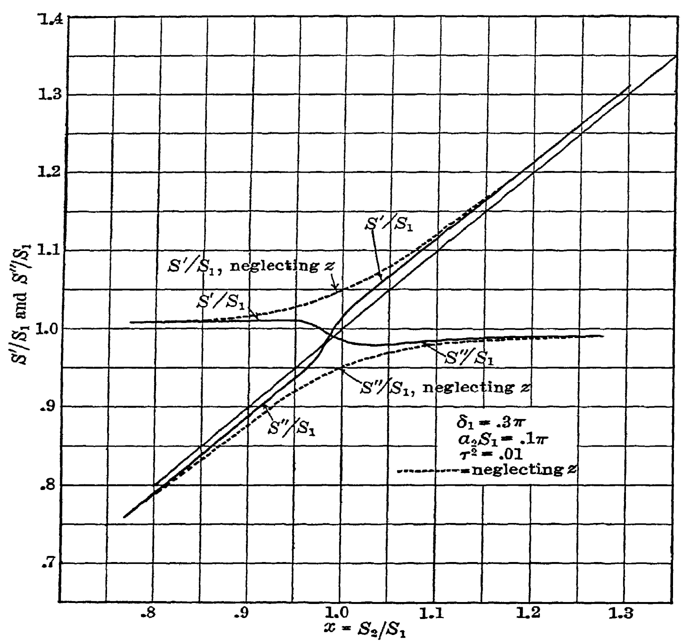

FIG. 8—Curves similar to Fig. 7, but with $\tau^2 = 0.01$, and with added dotted curves showing the effect of neglecting z.

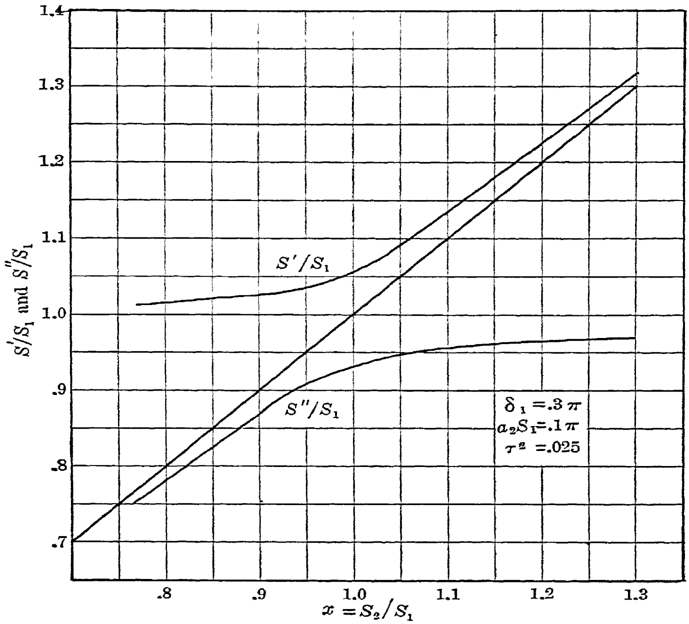

FIG. 9.—Curves of ratios of undamped periods for circuits not isochronous plotted against S_2/S_1, for $\tau^2 = 0.025$.

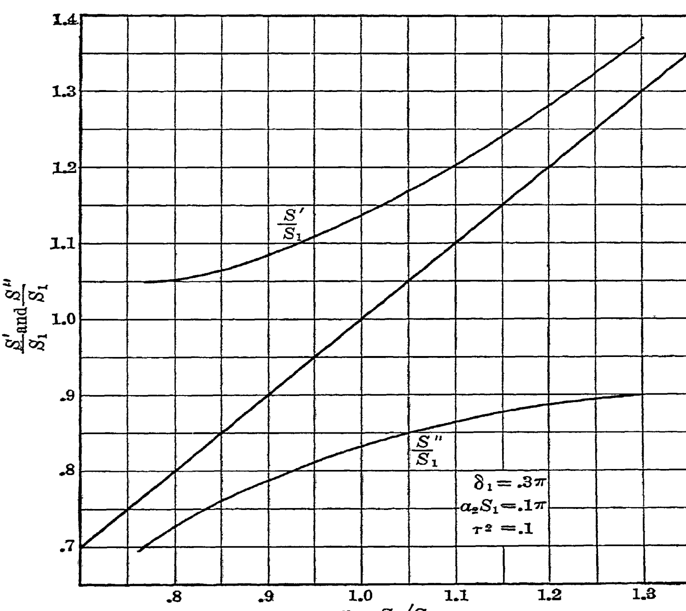

FIG. 10.—Same as Fig. 9 except that $\tau^2 = 0.1$.

and (49) are correct. We shall next examine (48) and (49) with a view to using them in the present numerical case.

Multiplying (48) through by S_1/π, and replacing a_1S_1 by 0.3π and a_2S_1 by 0.1π, we obtain

$$\frac{a'S_1}{\pi} = \frac{0.2}{1-\tau^2} - \frac{1}{2}\sqrt{\frac{0.16}{(1-\tau^2)^2} - \frac{4(0.03)}{1-\tau^2} + \frac{4z}{x(1-\tau^2)}}$$
$$= \frac{0.2}{1-\tau^2}\{1 - \sqrt{1-\varphi}\} \tag{122}$$

where

$$\varphi = \frac{(1-\tau^2)}{0.04}\left\{0.03 - \frac{z}{x}\right\}$$

In like manner, from (49) we obtain

$$\frac{a''S_1}{\pi} = \frac{0.2}{1-\tau^2}\{1 + \sqrt{1-\varphi}\} \tag{123}$$

In using these equations it is to be borne in mind that, if (121) *is not fulfilled, the signs before the radicals in* (122) *and* (123) *are to be interchanged.*

125. Criterion Values.—Applying the criterion inequality (121) to the present numerical cases it is found that the signs given in (122) and (123) are correct for all values of x greater than a certain limiting value for each value of τ^2. These limiting values are as follows:

τ^2	Limiting value of x Signs in (122) and (123) are correct for x greater than
0 001	0 999
0 010	0 987
0 025	0 961
0 100	0 805

Keeping these criterion values in mind, equations (122) and (123) were used in computation of the values of $a'S_1/\pi$ and $a''S_1/\pi$ recorded in Tables IV to VII, and plotted in the curves of Figs. 11 to 14.

126. Examination of Results in the General Case with Numerical Constants.—The results contained in Tables III to VII will now be examined. The given constants used in the comp tation of these tables are $\delta_1 = 0.3\pi$, $a_2S_1 = 0.1\pi$, while the coefficient of coupling had four different values whose squares are $\tau^2 = 0.001$, $\tau^2 = 0.01$, $\tau^2 = 0.025$, $\tau^2 = 0.1$.

127. Examination of z.—Table III contains values of $-z$ for various values of x ($=S_2/S_1$), and for the four different values of τ^2. These results are plotted in Fig. 6. It will be seen that in each case $-z$ has a maximum.

For the two smaller values of τ^2 (*i.e.*, for $\tau^2 = 0.001$ and $\tau^2 = 0.01$) the maximum value of $-z$ is approximately equal to τ^2, and this maximum value occurs at a value of x a little less than unity.

For the two larger values of τ^2, the maximum value of $-z$ is much smaller than τ^2 and occurs at a value of x considerably different from unity.

128. Examination of the Undamped Periods.—The values of the undamped periods, in the form of their ratios to S_1, are given in Tables IV to VII, for different values of x ($=S_2/S_1$) and for the different values of τ^2. Each of the tables corresponds to a particular value of τ^2.

In these tables the quantities in parentheses are the values that are obtained if we consider z to be zero, while the values not in parentheses are the values obtained by giving z its proper value, and taking account of its effect on the resultant periods.

A comparison of the values not in parentheses with those in parentheses shows the amount of the error that would be made in this numerical case of rather large damping if z were entirely neglected. The effect of the z differs with the coefficient of coupling τ and with the ratio x of the undamped periods of the original circuits.

From Table IV, in which $\tau^2 = 0.001$, it is seen that the effect of z is inappreciable for large and for small value of x (that is, for values in which the original circuits are widely out of synchronism), but at $x = 1$ (*i.e.*, with the circuits synchronous) the effect of z in this case is to modify the computed periods by about 1 per cent.

From Table V, in which $\tau^2 = 0.01$, it is seen that at $x = 0.98$ the effect of z is to modify the computed values by about 4 per cent. In this case also, the effect of z is hardly appreciable for large and for small values of x.

Table VI, with $\tau^2 = 0.025$, shows that the effect of z is to modify the computed periods by about 2 per cent. for x in the neighborhood of 1, with this effect decreasing toward the small values of x and almost inappreciable at the large values of x.

Similarly, Table VII, for $\tau^2 = 0.1$, shows that the effect of z

Table IV.—Computed Values Involving Damping Constants and Undamped Periods in the General Case with Various Values of $x = S_2/S_1$. Given $\delta = 0.3\pi$, $a_2S_1 = 0.1\pi$ and $\tau^2 = 0.001$

Values in parentheses are values obtained by regarding z as negligible.

$x = S_2/S_1$	$a'S_1/\pi$	$a''S_1/\pi$	S'/S_1	S''/S_1
0 76923	0 3000	0 1004	1 0007 (1 0009)	0 7683 (0 7682)
0 83333	0 2999	0 1005	1 0006 (1 0010)	0 8324 (0 8320)
0 90909	0 2980	0 1024	1 0007 (1 0022)	0 9080 (0 9066)
0 95238	0 2969	0 1035	1 0005 (1 0046)	0 9514 (0 9476)
0 96154	0 2966	0 1038	1 0001 (1 0053)	0 9609 (0 9559)
0 97087	0 2966	0 1039	1 0000 (1 0067)	0 9705 (0 9639)
0 98039	0 2966	0 1039	0 9992 (1 0083)	0 9807 (0 9718)
0 99010	0 2966	0 1038	1 0003 (1 0114)	0 9893 (0 9784)
1 00	0 1039	0 2965	0 9987 (1 0038)	0 9972 (0 9938)
1 01	0 1037	0 2967	1 0072 (1 0104)	1 0023 (0 9977)
1 02	0 1034	0 2970	1 0136 (1 0160)	1 0059 (1 0037)
1 03	0 1030	0 2974	1 0199 (1 0218)	1 0096 (1 0079)
1 04	0 1029	0 2978	1 0270 (1 0285)	1 0135 (1 0128)
1.05	0 1025	9 2981	1 0332 (1 0343)	1 0163 (1 0154)
1 10	0 1007	0 2997	1 1014 (1 1025)	0 9982 (0 9973)
1 20	0 0994	0 3010	1 2013 (1 2014)	0 9986 (0 9983)
1 30	0 0990	0 3014	1 3009 (1 3009)	0 9988 (0 9987)

Table V.—Same as Table IV, Except That $\tau^2 = 0.01$

$x = S_2/S_1$	$a'S_1/\pi$	$a''S_1/\pi$	S'/S_1	S''/S_1
0 76923	0 2900	0.1141	1.0052 (1 0077)	0 7614 (0 7610)
0 83333	0 2857	0 1183	1 0071 (1 0105)	0 8233 (0.8205)
0 90909	0 2705	0.1336	1 0089 (1 0194)	0 8966 (0 8874)
0 95238	0 2532	0.1508	1 0087 (1 0294)	0 9395 (0 9205)
0 96154	0 2472	0 1569	1 0073 (1 0322)	0 9499 (0 9270)
0 97087	0 2404	0 1637	1 0065 (1 0307)	0.9599 (0 9327)
0 98039	0 2116	0.1925	0.9960 (1.0395)	0 9795 (0 9386)
0 99010	0 1923	0 2118	0 9925 (1 0439)	0 9925 (0 9437)
1 00	0 1672	0 2368	1 0121 (1 0489)	0.9831 (0 9487)
1 01	0 1520	0 2520	1 0260 (1.0544)	0 9794 (0 9532)
1 02	0 1443	0 2598	1 0363 (1.0604)	0 9793 (0 9572)
1 03	0.1370	0 2670	1 0470 (1 0667)	0.9788 (0 9607)
1 04	0.1340	0 2701	1.0559 (1.0716)	0 9801 (0.9658)
1 05	0 1279	0 2762	1.0666 (1 0808)	0 9796 (0 9666)
1 10	0 1134	0 2907	1 1152 (1.1212)	0.9815 (0.9762)
1 20	0 1037	0 3003	1 2094 (1 2109)	0.9879 (0 9867)
1.30	0.1006	0 3031	1 3086 (1.3091)	0 9884 (0 9881)

Table VI.—Same as Table IV, Except That $\tau^2 = 0.025$

$x = S_2/S_1$	$a'S_1/\pi$	$a''S_1/\pi$	S'/S_1	S''/S_1
0 76923	0 2766	0 1336	1.0079 (1 0168)	0 7538 (0 7470)
0 83333	0 2653	0.1450	1 0175 (1 0242)	0 8087 (0 7983)
0 90909	0 2386	0 1716	1 0263 (1 0404)	0 8747 (0 8630)
0 95238	0 2142	0 1960	1.0358 (1 0542)	0 9080 (0 8920)
0 96154	0 2051	0 2051	1 0399 (1 0579)	0 9154 (0 8975)
0 97087	0 1969	0 2123	1.0422 (1 0619)	0 9198 (0 9028)
0 98039	0 1892	0 2210	1.0464 (1 0651)	0 9252 (0 9068)
0 99010	0 1802	0.2300	1 0512 (1 0711)	0 9293 (0 9128)
1 00	0 1719	0 2383	1.0573 (1 0762)	0 9339 (0 9176)
1 01	0 1640	0 2463	1.0639 (1 0818)	0.9375 (0 9219)
1 02	0 1570	0 2532	1 0708 (1 0876)	0.9405 (0.9260)
1 03	0 1488	0 2615	1 0787 (1 0937)	0 9429 (0 9298)
1 04	0 1449	0 2653	1 0861 (1 1014)	0.9457 (0 9346)
1 05	0 1399	0 2704	1 0941 (1.1069)	0 9477 (0 9367)
1 10	0 1222	0 2881	1 1362 (1.1437)	0 9559 (0 9465)
1 20	0 1070	0 3033	1 2265 (1 2291)	0 9662 (0 9642)
1 30	0 1019	0 3084	1 3208 (1.3217)	0 0719 (0.9713)

Table VII.—Same as Table IV, Except That $\tau^2 = 0.1$

$x = S_2/S_1$	$a'S_1/\pi$	$a''S_1/\pi$	S'/S_1	S''/S_1
0 76923	0 2346	0.2098	1.0481 (1 0552)	0.6963 (0 6916)
0 83333	0 2139	0.2304	1.0570 (1.0732)	0.7502 (0 7262)
0 90909	0 1855	0.2589	1.0917 (1.1013)	0 7901 (0 7833)
0 95238	0 1688	0.2756	1.1116 (1 1218)	0.8128 (0 8052)
0 96154	0 1655	0.2790	1.1162 (1.1263)	0.8172 (0 8099)
0 97087	0.1620	0.2824	1.1213 (1.1312)	0.8214 (0.8142)
0 98039	0.1586	0.2858	1 1266 (1 1363)	0.8256 (0.8186)
0 99010	0 1552	0.2892	1 1322 (1.1417)	0 8297 (0.8227)
1 00	0 1519	0.2926	1 1381 (1.1473)	0.8336 (0.8269)
1 01	0.1485	0.2960	1.1441 (1.1531)	0.8375 (0.8310)
1 02	0 1455	0.2990	1 1504 (1 1591)	0 8412 (0.8349)
1 03	0.1424	0.3020	1.1569 (1 1653)	0 8447 (0.8386)
1 04	0.1395	0.3049	1.1633 (1.1715)	0 8481 (0.8423)
1 05	0 1376	0.3068	1 1701 (1 1779)	0 8514 (0.8457)
1 10	0.1250	0.3195	1.2045 (1 2106)	0 8643 (0.8599)
1 20	0.1127	0.3317	1.2847 (1.2875)	0 8863 (0.8836)
1.30	0.1020	0.3424	1.3702 (1.3718)	0 9001 (0.8991)

on the periods is small for large values of x. The effect of z is about 1 per cent on computed periods for values of x between about 0.95 and 1.02 For small values of x the effect of z is smaller than in the neighborhood of $x = 1$, but is still considerable for the smallest value of x used in the computations.

It will be interesting to compare the effect of z, which is the effect of the damping constants, on the resultant undamped periods S' and S'', with the effect of the damping constants on the original periods T_1 and T_2 of the circuits if not coupled.

Let us note that for any oscillatory single circuit the undamped period and the free period have respectively the values

$$S = 2\pi/\Omega, \qquad T = 2\pi/\omega = 2\pi/\sqrt{\Omega^2 - a^2} \tag{124}$$

whence

$$\begin{aligned}\frac{S}{T} &= \sqrt{1 - a^2/\Omega^2} = \sqrt{1 - a^2S^2/4\pi^2} = \sqrt{1 - \delta^2/4\pi^2} \\ &= 1 - \delta^2/8\pi^2, \text{ approximately}\end{aligned} \tag{125}$$

Using the values of δ_1 pertaining to this numerical example (0.3π), we have

$$\frac{S_1}{T_1} = 1 - 0.011,$$

while if δ_1 were zero S_1 would be equal to T_1, so that the effect of the damping in this circuit alone is to modify its period by about 1 per cent. For the other circuit with the decrement δ_2, which is smaller, the effect would be less.

It appears, therefore, that in the coupled system, the effect of the decrements in modifying the periods is as much as four times as great as with a single circuit standing alone (compare Table V).

Let us refer now to the curves of Figs. 7 to 10. In these curves S'/S_1 and S''/S_1 are plotted as ordinates and x $(= S_2/S_1)$ is plotted as abscissæ. We have adhered to the convention that of the two quantities S' and S'', the greater shall be designated S'. In Figs. 7 and 8, for $\tau^2 = 0.001$ and $\tau^2 = 0.01$ respectively, the curves consist of two lines that cross; and the upper part of each of these lines has been designated S'/S_1 and the lower part S''/S_1 to conform to the convention that $S' > S''$. In Figs. 9 and 10, which are for $\tau^2 = 0.025$ and $\tau^2 = 0.01$ respectively, the two curves do not cross or touch, and the curves for S'/S_1 and S''/S_1 are widely separated. The curves in these cases of the

larger coefficients of coupling are very similar in character to the corresponding period curves in which the resistances were considered to be zero, as in the dotted curves of Fig. 8. The values in the present cases, as given in Tables VI and VII, in which the decrements are rather large, differ by as much as 2 per cent. from the values obtained by neglecting the resistances.

A criterion can be obtained theoretically that will determine in any particular case whether the curves of S' and S'' meet, as in Figs. 7 and 8, or do not meet, as in Figs. 9 and 10, but this investigation is here omitted.

129. Examination of Damping Constants.—Tables IV to VII contain values of $a'S_1/\pi$ and $a''S_1/\pi$ for various values of x $(=S_2/S_1)$ and for four values of τ^2, as indicated in the headings to the tables. Here, as always, a' is the damping constant in the coupled system belonging to the undamped period S', which is the larger of the resultant undamped periods, and a'' is the damping constant in the coupled system belonging to S'', which is the smaller of the resultant undamped periods.

Curves corresponding to these damping constants are plotted in Figs. 11 to 14, with x as abscissæ, and with $a'S_1/\pi$ and $a''S_1/\pi$ as ordinates.

In Fig. 11, which is for the case of $\tau^2 = 0.001$, it is seen that for a range of x extending nearly up to $x = 1$, $a'S_1/\pi$ is approximately equal to 0.3 (which is the value of $a_1S_1/\pi = \delta_1/\pi = 0.3$ in this numerical case). The same quantity is approximately equal to 0.1 (that is, approximately equal to a_2S_1/π) for a range of x extending from $x = 1$ on up to the largest value of x given. The curve of $a''S_1/\pi$ does the same thing over a reversed pair of ranges.

We may express this result as follows:

In this special case of $\tau^2 = 0.001$, we see that

I. If $x < 0.99$,

$$a' = a_1 \text{ and } a'' = a_2, \text{ approximately,}$$

II. If $x > 1$,

$$a' = a_2 \text{ and } a'' = a_1, \text{ approximately,}$$

III. Between $x = 0.99$ and $x = 1$, a' and a'' undergo transition,

IV. At $x = 0.995$ the damping constants a' and a'' are equal.

These simple relations are incident to the looseness of the coupling in this case.

The curves of Fig. 12 show that with the larger coefficient of

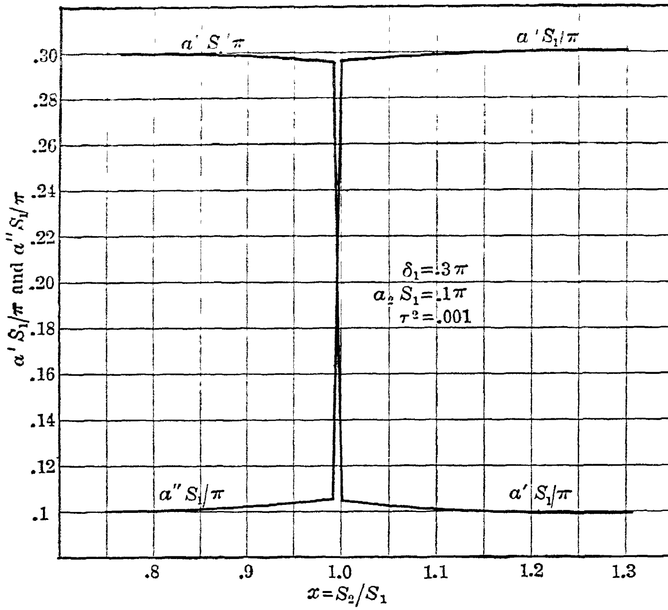

FIG 11.—Ratios involving resultant damping factors for nonisochronous circuits plotted against S_2/S_1 for $\delta_1 = 0.3\pi$, $a_2S_1 = 0.1\pi$, $\tau^2 = 0.001$.

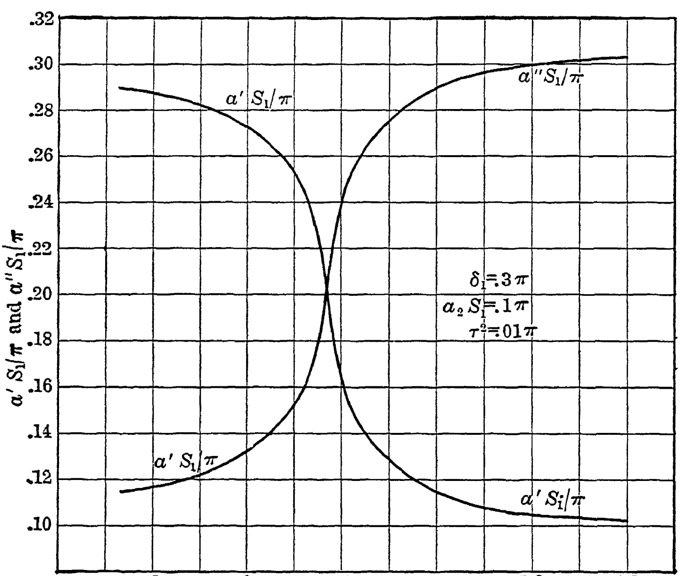

FIG. 12.—Same as Fig. 11, except that $\tau^2 = 0.01$.

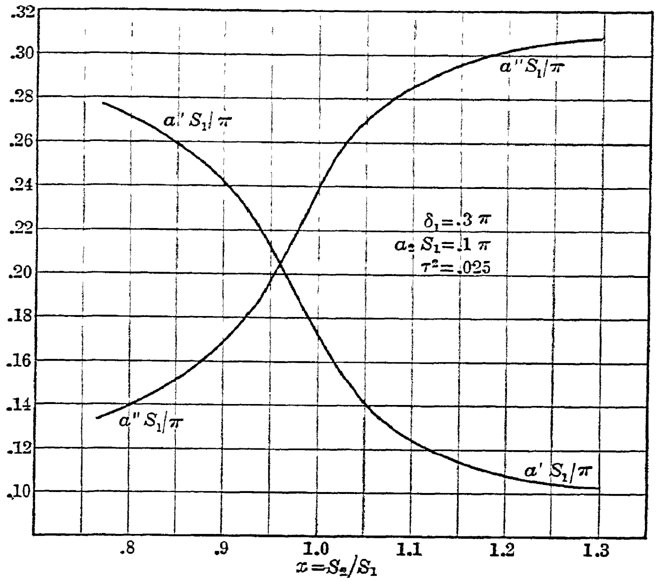

FIG. 13 —Same as Fig. 11, except that $\tau^2 = 0.025$.

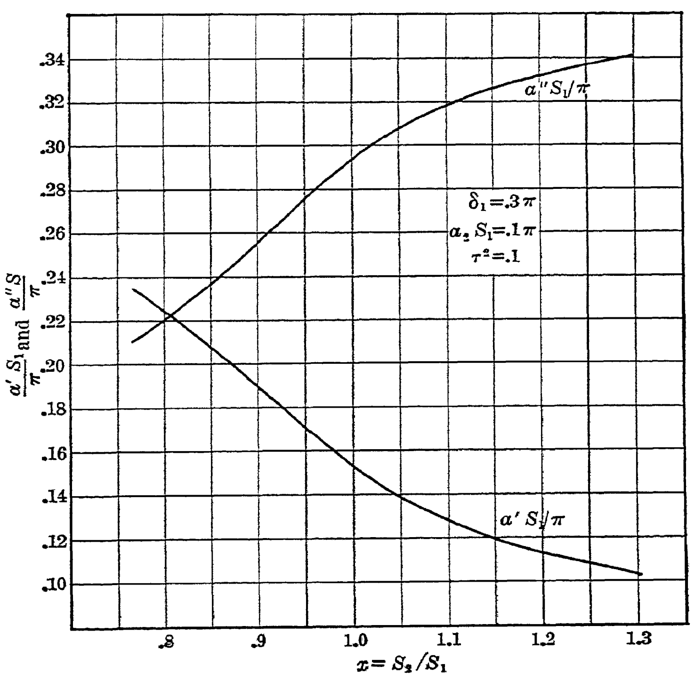

FIG. 14.—Same Fig. 11, except that $\tau^2 = 0.1$.

coupling (with $\tau^2 = 0.01$) the region of transition is spread out so that it embraces practically the whole plotted range of x The intersection point in this case is at about $x = 0.987$.

With still larger coefficients of coupling (with $\tau^2 = 0.025$ and $\tau^2 = 0.1$), the curves of Figs 13 and 14 show that the intersecting points are still further shifted toward the smaller values of x. These points appear at $x = 0.961$ and $x = 0.805$ respectively, as has been previously determined

III. THE LOOSE COUPLED SYSTEM

130. Determination of the Oscillation Constants When $\tau = 0$. We can best obtain the result in this case by letting $\tau^2 = 0$ in the original fourth degree equation (10), which then factors into

$$k^2 + 2a_1k + \Omega_1^2 = 0, \text{ or}$$

$$k^2 + 2a_2k + \Omega_2^2 = 0,$$

whence replacing either Ω^2 by $a^2 + \omega^2$, we have

$$k^2 + 2a_1k + a_1^2 = -\omega_1^2, \text{ or}$$

$$k^2 + 2a_2k + a_2^2 = -\omega_2^2.$$

Extracting the square roots we have

$$k = -a_1 \pm j\omega_1, \text{ or}$$

$$k = -a_2 \pm j\omega_2.$$

From the definitions of k given in (20) we see that

$$a' = a_1, \text{ or } a_2, \quad a'' = a_2, \text{ or } a_1 \tag{126}$$

$$\omega' = \omega_1, \text{ or } \omega_2, \quad \omega'' = \omega_2, \text{ or } \omega_1 \tag{127}$$

By a combination of these equations we obtain also

$$S' = S_1, \text{ or } S_2, \quad S'' = S_2, \text{ or } S_1 \tag{128}$$

The ambiguity of these values can be removed by use of the convention that S' is greater than S'', so that

$$\left.\begin{aligned} \text{if } S_1 > S_2, \quad a' = a_1, \quad \omega' = \omega_1, \quad S' = S_1 \\ a'' = a_2, \quad \omega'' = \omega_2, \quad S'' = S_2 \end{aligned}\right\} \tag{129}$$

while

$$\left.\begin{aligned} \text{if } S_1 < S_2, \quad a' = a_2, \quad \omega' = \omega_2, \quad S' = S_2 \\ a'' = a_1, \quad \omega'' = \omega_1, \quad S'' = S_1 \end{aligned}\right\} \tag{130}$$

For convenience we may also here write the values of the periods and wavelengths, as follows:

$$\text{if } S_1 > S_2,\ T' = T_1,\ T'' = T_2,\ \lambda' = \lambda_1,\ \lambda'' = \lambda_2 \qquad (131)$$

while

$$\text{if } S_1 < S_2,\ T' = T_2,\ T'' = T_1,\ \lambda' = \lambda_2,\ \lambda'' = \lambda_1 \qquad (132)$$

Equations (129) *to* (132) *give the values of the oscillation constants of the coupled system in terms of the values of the constants of the system not coupled, provided* τ^2 *is effectually zero.*

The next Chapter will treat of the amplitudes in the system of two magnetically coupled circuits.

CHAPTER X

AMPLITUDE AND MEAN SQUARE CURRENT IN THE INDUCTIVELY COUPLED SYSTEM OF TWO CIRCUITS

131. Continuation of Preceding Chapter.—In the preceding chapter the discussion was confined mainly to the periods and damping constants of the coupled system of two circuits related as shown in Fig. 1 of Chapter IX.

However, in equations (15) and (16) we have given the general expressions for the currents i_1 and i_2 in the two circuits respectively in the form

$$\left.\begin{aligned} i_1 &= \Sigma A_n \epsilon^{k_n t} \\ i_2 &= \Sigma B_n \epsilon^{k_n t} \end{aligned}\right\} \text{where } n = 1, 2, 3, 4 \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

We have found that the four k's, k_1, k_2, k_3 and k_4, are the four roots of a fourth degree algebraic equation, (19) Chapter IX, and that these roots may be written as two pairs of conjugate imaginary quantities as follows:

$$\begin{aligned} k_1 &= -a' + j\omega', \qquad & k_3 &= -a'' + j\omega'', \\ k_2 &= -a' - j\omega', \qquad & k_4 &= -a'' - j\omega''. \end{aligned}$$

In the preceding chapter the discussion of these roots was entered into at length. We propose now to return to the matter of the amplitudes of the primary and secondary currents, for two sets of initial conditions based on two modes of exciting the oscillation. These two methods are first, *Excitation by Discharging the Primary Condenser* and, second, *Excitation by Discharging the Primary Inductance.* These titles will constitute the major headings of the material of the present Chapter.

EXCITATION BY DISCHARGING THE PRIMARY CONDENSER

132. Initial Conditions with C_1 Initially Charged and Allowed to Discharge.—If the primary condenser C_1 is supposed to be initially charged with a quantity of electricity Q_1, while the secondary condenser is initially uncharged, and if the currents

in the primary and secondary circuits are initially zero, we shall have the initial conditions,

$$\left.\begin{aligned} t &= 0 \\ i_1 &= 0 \\ i_2 &= 0 \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} q_1 &= \int i_1 dt = Q_1 \\ q_2 &= \int i_2 dt = 0 \end{aligned}\right\} \tag{4}$$

These initial conditions when substituted into (1) and (2) give

$$\left.\begin{aligned} \Sigma A_n &= 0, & \Sigma B_n &= 0 \\ \Sigma\frac{A_n}{k_n} &= Q_1, & \Sigma\frac{B_n}{k_n} &= 0 \end{aligned}\right\} \tag{5}$$

where Σ denotes a summation extended to $n = 1, 2, 3, 4$.

133. Relations Among the A's and B's.—There are four relations among the A's and B's as given in equation (7) or (8) of the preceding chapter, for each of these equations for any given A, B, and k, all with the same subscript 1, 2, 3, or 4. The subscripts of the L, R, and C, are not to be permuted with permutation of the subscripts of A, B, and k.

The equations (7) and (8), when each is divided by k, may be written

$$L_1 A_n + R_1 \frac{A_n}{k_n} + \frac{1}{C_1}\frac{A_n}{k_n^2} = MB_n \tag{6}$$

$$L_2 B_n + R_2 \frac{B_n}{k_n} + \frac{1}{C_2}\frac{B_n}{k_n^2} = MA_n \tag{7}$$

We may now perform useful eliminations among equations of the type of (6) and (7) as follows:

If we take the sum of the four equations comprised in (6) when n is given respectively the values 1, 2, 3, and 4, we obtain

$$L_1 \Sigma A_n + R_1 \Sigma \frac{A_n}{k_n} + \frac{1}{C_1} \Sigma\frac{A_n}{k_n^2} = M\Sigma B_n \tag{8}$$

whence, by (5),

$$0 + R_1 Q_1 + \frac{1}{C_1} \Sigma\frac{A_n}{k_n^2} = 0 \tag{9}$$

and by transposition

$$\Sigma\frac{A_n}{k_n^2} = -\, C_1 R_1 Q_1 \tag{10}$$

A similar treatment of equation (7) gives

$$\Sigma\frac{B_n}{k_n^2} = 0 \tag{11}$$

If we next take equation (6), and divide it by k_n, we shall have four equations, corresponding respectively to $n = 1, 2, 3$, and 4. If now we add these four equations, we obtain

$$L_1\Sigma\frac{A_n}{k_n} + R_1\Sigma\frac{A_n}{k_n^2} + \frac{1}{C_1}\,\Sigma\frac{A_n}{k_n^3} = M\Sigma\frac{B_n}{k_n}.$$

which in view of (5) and (10) reduces to

$$L_1Q_1 - C_1R_1^2Q_1 + \frac{1}{C_1}\,\Sigma\frac{A_n}{k_n^3} = 0.$$

This, by transposition, gives

$$\Sigma\frac{A_n}{k_n^3} = C_1^2R_1^2Q_1 - L_1C_1Q_1 \tag{12}$$

A corresponding treatment of the four equations comprehended in equation (7) gives

$$\Sigma\frac{B_n}{k_n^3} = MC_2Q_1 \tag{13}$$

134. Summary of These Results.—With excitation of the system by discharging the primary condenser initially charged with a quantity Q_1 of electricity, the four A's and the four B's were found to satisfy the following group of equations

$$\left.\begin{array}{ll}\Sigma A_n = 0, & \Sigma B_n = 0\\[4pt] \Sigma\dfrac{A_n}{k_n} = Q_1 & \Sigma\dfrac{B_n}{k_n} = 0\\[4pt] \Sigma\dfrac{A_n}{k_n^2} = -\,C_1R_1Q_1, & \Sigma\dfrac{B_n}{k_n^2} = 0\\[4pt] \Sigma\dfrac{A_n}{k_n^3} = C_1^2R_1^2Q_1 - L_1C_1Q_1, & \Sigma\dfrac{B_n}{k_n^3} = MC_2Q_1\end{array}\right\} \tag{14}$$

135. Determination of the Values of B_1, B_2, B_3, and B_4.—The values of the several B's may be determined by solving the four simultaneous B-equations of (14). B_1 is thus found to be given by the determinant equations

$$B_1\begin{vmatrix}1 & 1 & 1 & 1\\ \dfrac{1}{k_1} & \dfrac{1}{k_2} & \dfrac{1}{k_3} & \dfrac{1}{k_4}\\ \dfrac{1}{k_1^2} & \dfrac{1}{k_2^2} & \dfrac{1}{k_3^2} & \dfrac{1}{k_4^2}\\ \dfrac{1}{k_1^3} & \dfrac{1}{k_2^3} & \dfrac{1}{k_3^3} & \dfrac{1}{k_4^3}\end{vmatrix} = -MC_2Q_1\begin{vmatrix}1 & 1 & 1\\ \dfrac{1}{k_2} & \dfrac{1}{k_3} & \dfrac{1}{k_4}\\ \dfrac{1}{k_2^2} & \dfrac{1}{k_3^2} & \dfrac{1}{k_4^2}\end{vmatrix} \tag{15}$$

Subtracting the first column of the left-hand determinant from each of the other columns, and casting out the first row and column, we obtain

$$B_1 \begin{vmatrix} \frac{1}{k_2} - \frac{1}{k_1} & \frac{1}{k_3} - \frac{1}{k_1} & \frac{1}{k_4} - \frac{1}{k_1} \\ \frac{1}{k_2^2} - \frac{1}{k_1^2} & \frac{1}{k_3^2} - \frac{1}{k_1^2} & \frac{1}{k_4^2} - \frac{1}{k_1^2} \\ \frac{1}{k_2^3} - \frac{1}{k_1^3} & \frac{1}{k_3^3} - \frac{1}{k_1^3} & \frac{1}{k_4^3} - \frac{1}{k_1^3} \end{vmatrix}$$

Factoring we obtain

$$B_1\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\left(\frac{1}{k_3} - \frac{1}{k_1}\right)\left(\frac{1}{k_4} - \frac{1}{k_1}\right) \times \tag{16}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{k_2} + \frac{1}{k_1} & \frac{1}{k_3} + \frac{1}{k_1} & \frac{1}{k_4} + \frac{1}{k_1} \\ \frac{1}{k_2^2} + \frac{1}{k_2k_1} + \frac{1}{k_1^2} & \frac{1}{k_3^2} + \frac{1}{k_3k_1} + \frac{1}{k_1^2} & \frac{1}{k_4^2} + \frac{1}{k_4k_1} + \frac{1}{k_1^2} \end{vmatrix}$$

The determinant of this last expression may be simplified by subtracting $1/k_1$ times the second row from the third row, and $1/k_1$ times the first row from the second, giving for the determinant factor alone

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{k_2} & \frac{1}{k_3} & \frac{1}{k_4} \\ \frac{1}{k_2^2} & \frac{1}{k_3^2} & \frac{1}{k_4^2} \end{vmatrix}$$

This determinant is to be multiplied by the factors before the multiplication sign in (16) and equated to the right-hand side of (15), and gives as the resultant equation

$$B_1 = -\frac{MC_2Q_1k_1^3k_2k_3k_4}{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)} \tag{17}$$

In a similar manner we may obtain

$$B_2 = -\frac{MC_2Q_1k_2^3k_1k_3k_4}{(k_2 - k_1)(k_2 - k_3)(k_2 - k_4)} \tag{18}$$

$$B_3 = -\frac{MC_2Q_1k_3^3k_1k_2k_4}{(k_3 - k_1)(k_3 - k_2)(k_3 - k_4)} \tag{19}$$

$$B_4 = -\frac{MC_2Q_1k_4^3k_1k_2k_3}{(k_4 - k_1)(k_4 - k_2)(k_4 - k_3)} \tag{20}$$

Here it is seen that with the given set of initial conditions the B's are completely determined in terms of the values of the k's. The values of A_1, A_2, A_3, and A_4, can be obtained from the corresponding values of the B's by use of the equations (7). We shall, however, not write out the values of the A's, but shall continue an investigation of the B's.[1]

136. Determination of i_2 in Trigonometric Form.—If now we introduce into the equation for B_1 the value immediately following equation(2), of the k's in terms of a's and ω's, and if we write, as usual,

$$\Omega'^2 = a'^2 + \omega'^2, \qquad \Omega''^2 = a''^2 + \omega''^2,$$

we obtain

$$B_1 = -MC_2Q_1 \left[\frac{\Omega'^2\Omega''^2(-a' + j\omega')^2}{2j\omega'\{a'' - a' + j(\omega' - \omega'')\}\{a'' - a' + j(\omega' + \omega'')\}}\right] \tag{21}$$

Now recalling that a complex quantity may be written in the exponential form, with the type

$$x + jy = \sqrt{x^2 + y^2}\, \epsilon^{j \tan^{-1}(y/x)},$$

and letting

$$H = \frac{MC_2Q_1\Omega'^2\Omega''^2}{\sqrt{\{(a'' - a')^2 + (\omega' - \omega'')^2\}\{(a'' - a')^2 + (\omega' + \omega'')^2\}}} \tag{22}$$

and

$$\varphi_1 = 2\tan^{-1}\frac{\omega'}{-a'} - \tan^{-1}\frac{\omega' - \omega''}{a'' - a'} - \tan^{-1}\frac{\omega' + \omega''}{a'' - a'} \tag{23}$$

we obtain

$$B_1 = -\frac{H\Omega'^2}{2j\omega'}\epsilon^{j\varphi_1} \tag{23a}$$

If now we treat B_2 in the same way, we shall find that B_2 differs from B_1 only in that the ω' has a different sign. It will be seen that this changes the sign of the φ_1, and also changes the sign of the whole quantity, since ω' enters as a divisor. That is,

$$B_2 = +\frac{H\Omega'^2}{2j\omega'}\epsilon^{-j\varphi_1} \tag{24}$$

[1] Up to here this chapter follows more or less closely P. Drude, *Ann. d. Phys.*, 25, p. 512, 1908. At this point I depart from Drude to avoid an error he makes in that on p. 531 in taking a time derivative of his equation (51) he overlooks the fact that B is a function of the time. The same error is made by Bjerknes before Drude and persists through much of the literature.

A similar treatment of B_3 and B_4 gives

$$B_3 = -\frac{H\Omega''^2}{2j\omega''}\epsilon^{+j\varphi_2} \tag{25}$$

$$B_4 = +\frac{H\Omega''^2}{2j\omega''}\epsilon^{-j\varphi_2} \tag{26}$$

in which

$$\varphi_2 = 2\tan^{-1}\frac{\omega''}{-a''} - \tan^{-1}\frac{\omega' - \omega''}{a'' - a'} + \tan^{-1}\frac{\omega' + \omega''}{a'' - a'} \tag{27}$$

If now we introduce these quantities into equation (2) for i_2, we shall have

$$\begin{aligned} i_2 = &- \frac{H\Omega'^2}{\omega'}\epsilon^{-a't}\left\{\frac{\epsilon^{j(\omega't+\varphi_1)} - \epsilon^{-j(\omega't+\varphi_1)}}{2j}\right\} \\ &- \frac{H\Omega''^2}{\omega''}\epsilon^{-a''t}\left\{\frac{\epsilon^{j(\omega''t+\varphi_2)} - \epsilon^{-j(\omega''t+\varphi_2)}}{2j}\right\} \end{aligned} \tag{28}$$

This equation becomes in trigonometric form

$$i_2 = \frac{-H\Omega'^2}{\omega'}\epsilon^{-a't}\sin(\omega't + \varphi_1) - \frac{H\Omega''^2}{\omega''}\epsilon^{-a''t}\sin(\omega''t + \varphi_2) \tag{29}$$

where H, φ_1, and φ_2 are defined in equations (22), (23), and (27).

Equation (29) gives the exact value of the current i_2 in the secondary circuit of the coupled system produced by the discharge of the condenser C_1 in the primary circuit. The condenser C_1 was initially charged with a quantity of electricity Q_1.

137. Integral Effect in Secondary Circuit.—If the secondary circuit contains a hot-wire ammeter or other instrument that is affected proportionally to the square of the current, it becomes important to obtain the value of the time integral of the square of the current extended over the time of one complete discharge. If we call this integral J, then by direct integration of the square of (29), we obtain

Term No.

$$\begin{aligned} J = &\int_0^\infty i_2^2 dt \\ = &H^2\left\{\frac{\Omega'^4}{4\omega'^2a'} + \frac{\Omega''^4}{4\omega''^2a''}\right. && (1) \\ &+ \frac{\Omega'^3}{4\omega'^2}\cos\left(2\varphi_1 - \tan^{-1}\frac{\omega'}{-a'}\right) && (2) \\ &+ \frac{\Omega''^3}{4\omega''^2}\cos\left(2\varphi_2 - \tan^{-1}\frac{\omega''}{-a''}\right) && (3) \\ &- \frac{\Omega'^2\Omega''^2}{\omega'\omega''}\left[\frac{\cos\left(\varphi_1 - \varphi_2 - \tan^{-1}\frac{\omega' - \omega''}{-(a' + a'')}\right)}{\sqrt{(a' + a'')^2 + (\omega' - \omega'')^2}}\right. && (4) \\ &\left.\left. - \frac{\cos\left(\varphi_1 + \varphi_2 - \tan^{-1}\frac{\omega' + \omega''}{-(a' + a'')}\right)}{\sqrt{(a' + a'')^2 + (\omega' + \omega'')^2}}\right]\right\} && (5) \quad (30) \end{aligned}$$

The terms of this equation are numbered for future reference.

Equation (30) *is exact. It gives the integral of the square of the secondary current produced by the discharge of a condenser in the primary circuit, in terms of the resultant damping constants and angular velocities.*

We shall next abandon strict accuracy and see how to replace equation (30) by an approximation suitable for calculation in certain important cases.

138. Approximate Treatment of the Integral Effect with Neglect of a^2 in Comparison with ω^2.—As an approximation to the value of J, let us first neglect all of the squares of the a's and the product of two a's in comparison with the squares of the ω's or in comparison with the product of two ω's, except where there appears differences of the squares of the ω's.

The term marked (1), within the brace of (30), is of the order $\omega^2/2a$. The coefficients, within the brace, of the trigonometric quantities that occur in the other terms have order as follows:

Table of Order of Coefficients of the Trigonometric Quantities in Various Terms of (30)

Term No	Order	Order relative to the order of term (1)
(2)	$\omega/4$	$a/2\omega$
(3)	$\omega/4$	$a/2\omega$
(4)	$\omega^2/2a$[1]	1
(5)	$\omega/2$	a/ω

We shall next examine the trigonometric quantities by which these coefficients are to be multiplied.

Let us call the trigonometric quantities in terms (2), (3), (4), and (5) respectively, F_2, F_3, F_4, and F_5. In these trigonometric quantities we shall expand the antitangents of those quantities known to be large (that is, of the order of ω/a) by the well known formula

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \quad . \quad . \quad . \quad , \text{where } x^2 > 1 \qquad (31)$$

and shall neglect terms of the order of a^3/ω^3 in comparison with a/ω.

[1] In the extreme case in which $(\omega' - \omega'')^2$ happens to be negligible in comparison with $2(a' + a'')^2$ To cover all contingencies we estimate its

This gives

$$F_2 = \sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'} - \frac{3a'}{\omega'} - \frac{2(a'' - a')}{\omega' + \omega''}\right],$$

$$F_3 = \sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'} - \frac{3a''}{\omega''} + \frac{2(a'' - a')}{\omega' + \omega''}\right],$$

$$F_4 = -\cos\left[\frac{2a'}{\omega'} - \frac{2a''}{\omega''} + \frac{2(a'' - a')}{\omega' + \omega''} - \tan^{-1}\frac{\omega' - \omega''}{-(a'' + a')}\right],$$

$$F_5 = -\sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'} + \frac{a' + a''}{\omega' + \omega''} - \frac{2a'}{\omega'} - \frac{2a''}{\omega''}\right].$$

Let us now continue our omission of squares or higher powers of a/ω in comparison with unity, and expand the above expressions, with replacement of $\sin(a/\omega)$ by a/ω and $\cos(a/\omega)$ by unity. This process gives

$$F_2 = \sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right] - \left[\frac{3a'}{\omega'} + \frac{2(a'' - a')}{\omega' + \omega''}\right]\cos\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right],$$

$$F_3 = \sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right] - \left[\frac{3a''}{\omega''} - \frac{2(a'' - a')}{\omega' + \omega''}\right]\cos\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right],$$

$$F_4 = -\cos\left[\tan^{-1}\frac{\omega' - \omega''}{-(a'' + a')}\right] - \left[\frac{2a'}{\omega'} - \frac{2a''}{\omega''} + \frac{2(a'' - a')}{\omega' + \omega''}\right]\sin\left[\tan^{-1}\frac{\omega' - \omega''}{-(a'' + a')}\right],$$

$$F_5 = -\sin\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right] + \left[\frac{2a'}{\omega'} + \frac{2a''}{\omega''} - \frac{a' + a''}{\omega' + \omega''}\right]\cos\left[2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right].$$

The cosine terms in the expressions for F_2, F_3, and F_5 have as multipliers quantities of the order of a/ω, and since the coefficients by which these F's are to be multiplied in forming J (see Table of Coefficients) are of the relative order of a/ω, these cosine terms will be neglected leaving

$$-F_5 = F_3 = F_2 = \sin\left(2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right)$$

$$= \frac{2(a'' - a')(\omega' - \omega'')}{(a'' - a')^2 + (\omega' - \omega'')^2} \qquad (32)$$

The remaining F, F_4, written out by the formulas for the sine and cosine of an antitangent, gives

$$F_4 = \frac{a'' + a' - \left[\frac{2a'}{\omega'} - \frac{2a''}{\omega''} + \frac{2(a'' - a')}{\omega' + \omega''}\right](\omega' - \omega'')}{\sqrt{(a'' + a')^2 + (\omega' - \omega'')^2}}$$

$$= \frac{a'' + a' - \frac{(\omega' - \omega'')}{(\omega' + \omega'')}\left\{\frac{2(a'\omega''^2 - a''\omega'^2)}{\omega'\omega''}\right\}}{\sqrt{(a'' + a')^2 + (\omega' - \omega'')^2}} \qquad (33)$$

If now we introduce these several results into (30) and at the same time replace the Ω's by ω's, we obtain

Term No

$$J = H^2\left\{\frac{\omega'^2}{4a'} + \frac{\omega''^2}{4a''}\right. \qquad (1)$$

$$+ \frac{(\omega' - \omega'')^2}{2(\omega' + \omega'')}\left[\frac{(a'' - a')(\omega' - \omega'')}{(a'' - a')^2 + (\omega' - \omega'')^2}\right] \qquad (2)$$

$$\left.-\left[\frac{\omega'\omega''(a'' + a') - 2(a'\omega''^2 - a''\omega'^2)\frac{\omega' - \omega''}{\omega' + \omega''}}{(a'' + a')^2 + (\omega' - \omega'')^2}\right]\right\} \qquad (3) \quad (34)$$

where

$$J = \int_0^\infty i_2{}^2 dt,$$

$$H = \frac{MC_2Q_1\omega'^2\omega''^2}{(\omega' + \omega'')\sqrt{(a'' - a')^2 + (\omega' - \omega'')^2}} \qquad (35)$$

Equation (34) for J gives the integral of the square of the secondary current for a complete discharge under the condition that the square of each of the damping constants a is negligible in comparison with the square of the angular velocities ω. No other approximation has been made. The result is in terms of the damping constants and angular velocities of the coupled system.

139. Value of the Integral of the Square of the Secondary Current for Two Circuits of Small Damping, Nearly in Resonance and Very Loosely Coupled.—Under the conditions given in this caption, the expression for the time integral of the square of the current in the secondary circuit reduces to a simple form. Assumptions are to be made as follows:

Assumption I.—The damping constants are supposed to be so small that their squares are negligible in comparison with the squares of the angular velocities. This assumption is fulfilled

by circuits even when the damping constants are large enough to cut the amplitude of current to one-half in one oscillation. The introduction of this assumption permits the use of equation (34) for J.

Assumption II.—The coefficient of coupling is supposed to be so small that we may with close approximation take

$$\left.\begin{array}{l} a' = a_1,\ a'' = a_2,\ \omega' = \omega_1,\ \omega'' = \omega_2 \\ \text{or} \\ a' = a_2,\ a'' = a_1,\ \omega' = \omega_2,\ \omega'' = \omega_1 \end{array}\right\} \quad (36)$$

as in equations (129) and (130), Chapter IX.

Assumption III.—The two circuits are assumed to be nearly in resonance so that ω_2 is nearly equal to ω_1, and, except in difference terms we shall replace ω_1^2, ω_2^2 and $\omega_1\omega_2$ by a common quantity ω^2. Also we shall assume

$$\omega_2 - \omega_1 << 2\omega \quad (37)$$

Referring now to equation (34) it is seen that these assumptions make the term marked "Term No. (2)" negligible in comparison with the term No. (1), since the quantity in the square bracket in No. (2) cannot be greater than ½.

Also it is seen that in Term No. (3) the term in the numerator subtracted from $\omega'\omega''\ (a'' + a')$ is negligible.

In the remaining terms, making the substitutions called for in Assumption II, we obtain from (34) the following simplified approximate value of J:

$$J = \frac{H^2\omega^2}{4}\left\{\frac{1}{a_1} + \frac{1}{a_2} - \frac{4(a_1 + a_2)}{(a_1 + a_2)^2 + (\omega_1 - \omega_2)^2}\right\} \quad (38)$$

where

$$H = \frac{MC_1Q_1\omega^3}{2\sqrt{(a_2 - a_1)^2 + (\omega_1 - \omega_2)^2}} \quad (39)$$

Equation (38) reduces to

$$J = \frac{H^2\omega^2(a_1 + a_2)}{4a_1a_2}\left\{\frac{(a_1 - a_2)^2 + (\omega_1 - \omega_2)^2}{(a_1 + a_2)^2 + (\omega_1 - \omega_2)^2}\right\} \quad (40)$$

Substituting for H^2 its value from (39), we obtain

$$J = \frac{M^2C_2^2Q_1^2\omega^8}{16a_1a_2} \qquad \frac{a_1 + a_2}{(a_1 + a_2)^2 + (\omega_1 - \omega_2)^2} \quad (41)$$

Equation (41) *gives the value of J* (*which is defined as*

$$J = \int_0^\infty i_2^2dt)$$

in the case of a secondary circuit very loosely coupled to a primary circuit, when the condenser in the primary is charged with a quantity of electricity Q_1 and allowed to discharge. The two circuits are supposed to have damping constants whose squares are negligible in comparison with the squares of the angular velocities, and the circuits are supposed to be not more than 5 or 10 per cent. out of resonance

The next section shows a method of using (41) to obtain the decrement of an unknown circuit.

140. Determination of the Decrement of an Unknown Circuit by Measuring the Integral Square Current in a Secondary Circuit Loosely Coupled with the Unknown Circuit.—One of the usual methods of measuring the logarithmic decrement d_1 of an oscillatory circuit is repeatedly to charge and discharge the con-

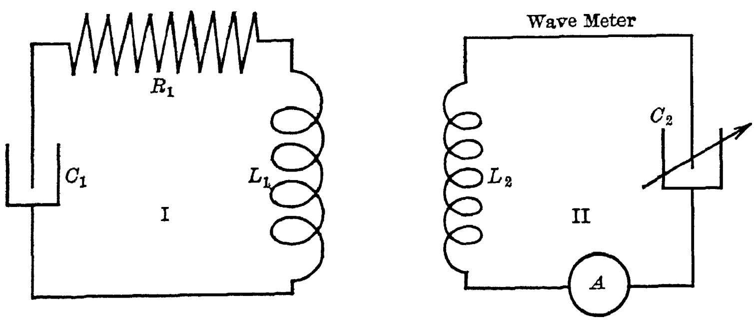

FIG. 1—For determining decrement of circuit I. Circuit II is a wavemeter with variable condenser C_2, and a current-measuring instrument at A.

denser C_1 (Fig. 1), or inductance L_1 of the given circuit, and to make wavelength measurements and integral square current measurements in a loosely coupled standard secondary circuit (II) of small decrement d_2. The standard circuit is usually a *wavemeter*, or, if calibrated to read directly in decrements, a *decremeter*.

The approximate formulas for obtaining decrements by this method are derivable from (41). If we call the value of J when $\omega_2 = \omega_1$ the *resonant value* of J, indicated by J_r, we have from (41)

$$J_r = \frac{M^2 C_2{}^2 Q_1{}^2 \omega^8}{16 a_1 a_2 (a_1 + a_2)} \tag{42}$$

By dividing (41) by (42), we obtain

$$y = \frac{J}{J_r} = \frac{(a_1 + a_2)^2}{(a_1 + a_2)^2 + (\omega_1 - \omega_2)^2}$$

$$= \frac{1}{1 + \dfrac{\omega_1{}^2}{(a_1 + a_2)^2}\left\{1 - \dfrac{\omega_2}{\omega_1}\right\}^2} \tag{43}$$

Let us now recall that

$$d_1 = a_1 T_1 = \frac{2\pi a_1}{\omega_1},$$

and at the same frequency

$$d_2 = a_2 T_1 = \frac{2\pi a_2}{\omega_1},$$

also

$$\omega_2/\omega_1 = \lambda_r/\lambda, \text{ where}$$

λ_r = the wavelength setting of the wavemeter at resonance,
λ = its wavelength setting for the reading J.

In terms of these quantities equation (43) becomes

$$y = \frac{1}{1 + \dfrac{4\pi^2}{(d_1 + d_2)^2}\left\{1 - \dfrac{\lambda_r}{\lambda}\right\}^2};$$

whence

$$d_1 + d_2 = \pm \frac{2\pi\left\{1 - \dfrac{\lambda_r}{\lambda}\right\}}{\sqrt{\dfrac{1 - y}{y}}} = \pm \frac{2\pi\left\{1 - \dfrac{\lambda_r}{\lambda}\right\}}{\sqrt{\dfrac{J_r}{J} - 1}} \tag{44}$$

in which that sign before the radical is to be taken that makes $d_1 + d_2$ positive.

A simple way of applying the formula is as follows: Plot a resonance curve of J against λ, as is illustrated in Fig. 2. Then if we take the two values of λ (λ_a and λ_b say) that give the same value of J, and call

$$\lambda_a - \lambda_b = \Delta\lambda \tag{45}$$

we shall have from (44)

$$d_1 + d_2 = + \frac{2\pi\left\{1 - \dfrac{\lambda_r}{\lambda_a}\right\}}{\sqrt{\dfrac{J_r}{J} - 1}}$$

and

$$d_1 + d_2 = - \frac{2\pi\left\{1 - \dfrac{\lambda_r}{\lambda_b}\right\}}{\sqrt{\dfrac{J_r}{J} - 1}},$$

whence by addition and division by 2,

$$d_1 + d_2 = \frac{\pi \left\{ \dfrac{\lambda_r}{\lambda_b} - \dfrac{\lambda_r}{\lambda_a} \right\}}{\sqrt{\dfrac{J_r}{J} - 1}},$$

and since $\lambda_a \lambda_b = \lambda_r^2$ approximately, we may write

$$d_1 + d_2 = \frac{\pi \Delta\lambda}{\lambda_r} \sqrt{\frac{J}{J_r - J}} \tag{46}$$

That is, to obtain $d_1 + d_2$, we take the width $\Delta\lambda$ of the resonance curve, Fig. 2, in meters wavelength at any height J, divided by the resonant wavelength λ_r in meters and multiply by π and by the square root of $J/(J_r - J)$.

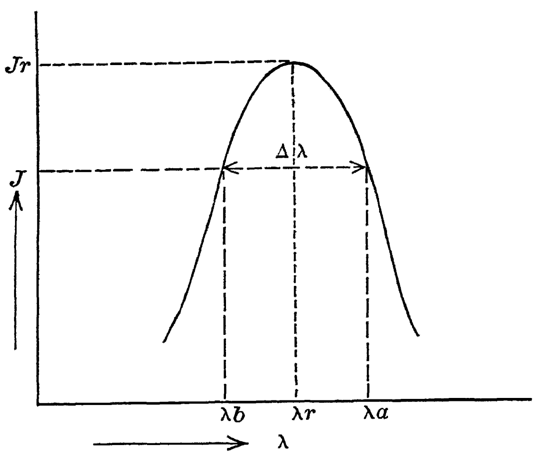

FIG 2.—Illustrating equation (46).

This formula is particularly easy to apply at the point where $J = J_r/2$, for the formula then becomes

$$d_1 + d_2 = \frac{\pi \Delta\lambda_h}{\lambda_r} \tag{47}$$

where $\Delta\lambda_h$ = the difference of the two wavelengths that give J one-half of its resonance value.

EXCITATION BY DISCHARGING THE PRIMARY INDUCTANCE

141. Initial Conditions When the Current is Produced by the Discharge of an Inductance in the Primary Circuit.—As has been pointed out in Chapter II, it is the practice in many

electrical measurements and in some small transmitting stations to excite the current oscillations by isolating a current in the primary inductance and allowing the current to subside. We have referred to this method of excitation as excitation by the discharge of an inductance.

The discharge of the inductance is effected in practice by the use of an electromagnetically driven interrupter as shown at J in Fig. 3, where is illustrated a coupled system operated in this way

A current from the battery B is sent through the inductance L_1, and when this current has a certain value I_1, which is practically steady, the feed current is opened at J.

We have then the initial conditions

$$\left.\begin{array}{ll} \text{when } t = 0, \quad i_1 = I_1, & i_2 = 0 \\ \qquad\qquad q_1 = -C_1R_1I_1, & q_2 = 0 \end{array}\right\} \tag{48}$$

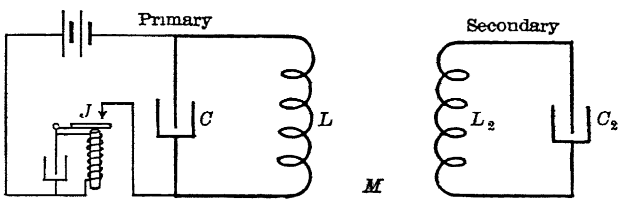

FIG. 3.

These conditions, so far as they pertain to a single circuit are discussed in Chapter II.

With these initial conditions, we are now to determine the values of A_n and B_n in the equations

$$i_1 = \Sigma A_n \epsilon^{k_n t}, \quad i_2 = \Sigma B_n \epsilon^{k_n t} \tag{49}$$

By integration of (49) we obtain

$$q_1 = \Sigma \frac{A_n \epsilon^{k_n t}}{k_n}, q_2 = \Sigma \frac{B_n \epsilon^{k_n t}}{k_n} \tag{50}$$

Now introducing the initial conditions (48) we obtain

$$\left.\begin{array}{ll} \Sigma A_n = I_1, & \Sigma B_n = 0 \\ \Sigma A_n/k_n = -C_1R_1I_1, & \Sigma B_n/k_n = 0 \end{array}\right\} \tag{51}$$

142. Manipulation of the Initial Conditions.—To obtain further relations concerning A and B, we shall make use of the equations (6) and (7). If in (7) we make n successively 1, 2, 3, 4, we obtain four équations, which added together give

$$L_2\Sigma B_n + R_2\Sigma \frac{B_n}{k_n} + \frac{1}{C_2}\Sigma \frac{B_n}{k_n^2} = M\Sigma A_n.$$

This equation, by (51), reduces to

$$\Sigma \frac{B_n}{k_n{}^2} = MC_2I_1 \tag{52}$$

which is a new equation in terms of B_η.

Let us now take equation (6), multiply each term by k_n, and sum up for the four k's; and let us perform a similar operation on (7). These two operations yield

$$L_1\Sigma A_nk_n + R_1\Sigma A_n + \frac{1}{C_1}\Sigma\frac{A_n}{k_n} = M\Sigma B_nk_n,$$

and

$$L_2\Sigma B_nk_n + R_2\Sigma B_n + \frac{1}{C_2}\Sigma\frac{B_n}{k_n} = M\Sigma A_nk_n.$$

By (51) these two equations reduce to

$$L_1\Sigma A_nk_n = M\Sigma B_nk_n, \qquad L_2\Sigma B_nk_n = M\Sigma A_nk_n \tag{53}$$

Solving the two equations of (53) as simultaneous, we obtain

$$\Sigma B_nk_n = 0, \qquad \Sigma A_nk_n = 0 \tag{54}$$

Collecting results, so far as concerns B, we have

$$\Sigma B_nk_n = 0,\ \Sigma B_n = 0,\ \Sigma B_n/k_n = 0,\ \Sigma B_n/k_n{}^2 = MC_2I_1 \tag{55}$$

It will not be necessary to go through the detail of solving these four simultaneous equations, as we can obtain the result by a direct comparison of these equations (55) with the corresponding equations (14) obtained with the condenser-discharge method of excitation. If in (55) we let $B_1 = Y_1/k_1$, $B_2 = Y_2/k_2$, etc., equations (55) in terms of Y_n will be of the same form as (14), with only the Q_1 of (14) replaced by I_1.

It thus appears that if we substitute I_1 for Q_1 in the values of B_n given in (17) to (20), and divide the result by k_n, we shall get B_n of the present problem.

This gives

$$B_1 = -\frac{MC_2I_1k_1{}^2k_2k_3k_4}{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)} \tag{56}$$

The other quantities B_2, B_3, B_4 can be obtained from (56) by advancing the subscripts of the k's.

In order now to put our result into trigonometric form we may take the result (23) of the previous problem, multiply it by I_1 and divide it by Q_1k_1, and, since

$$k_1 = \Omega'\epsilon^{j\tan^{-1}(\omega'/-\alpha')},$$

obtain for the present B_1,

$$B_1 = -\frac{I_1 H \Omega'}{2j\omega' Q_1} \epsilon^{j\left(\varphi_1 - \tan^{-1}\frac{\omega'}{-a'}\right)} \tag{57}$$

A similar treatment of the other B's, and their combination to form i_2, gives

$$i_2 = -\frac{I_1 H}{Q_1}\left\{\frac{\Omega'}{\omega'}\epsilon^{-a't}\sin\left(\omega' t + \varphi_1 - \tan^{-1}\frac{\omega'}{-a'}\right) + \frac{\Omega''}{\omega''}\epsilon^{-a''t}\sin\left(\omega'' t + \varphi_2 - \tan^{-1}\frac{\omega''}{-a''}\right)\right\} \tag{58}$$

In this equation H, φ_1 and φ_2 have the values given in (22), (23), and (27) respectively. The H that occurs in (58) is taken from the case of condenser-discharge method of excitation and contains Q_1, but this Q_1 is eliminated by the Q_1 of the denominator of (58). The Q_1 has no meaning in the present problem.

Equation (58) *gives the exact value of the current* i_2 *in the secondary of the coupled system when the system is excited by the discharge of the primary inductance originally traversed by a current* I_1. *The amplitudes are seen to be absolutely and relatively different from the corresponding amplitudes produced by excitation by condenser discharge (compare* (29)). *The phase of the current components is also changed from the previous case. The* Q_1 *occurring in the demoninator of* (58) *has no meaning and is eliminated by a* Q_1 *involved in the numerator in* H.

143. Value of the Integral of the Square of the Secondary Current in the Coupled System Excited by the Discharge of the Primary Inductance.—By making suitable changes in (30) we obtain in this case

$$J = \frac{I_1^2 H^2}{Q_1^2}\left\{\frac{\Omega'^2}{4\omega'^2 a'} + \frac{\Omega''^2}{4\omega''^2 a''} + \frac{\Omega'}{4\omega'^2}\cos\left(2\varphi_1 - 3\tan^{-1}\frac{\omega'}{-a'}\right) + \frac{\Omega''}{4\omega''^2}\cos\left(2\varphi_2 - 3\tan^{-1}\frac{\omega''}{-a''}\right)\right.$$
$$\left. - \frac{\Omega'\Omega''}{\omega'\omega''}\left[\frac{\cos\left(\varphi_1 - \varphi_2 - \tan^{-1}\frac{\omega'}{-a'} + \tan^{-1}\frac{\omega''}{-a''} - \tan^{-1}\frac{\omega' - \omega''}{-(a'+a'')}\right)}{\sqrt{(a'+a'')^2 + (\omega' - \omega'')^2}} - \frac{\cos\left(\varphi_1 + \varphi_2 - \tan^{-1}\frac{\omega'}{-a'} + \tan^{-1}\frac{\omega''}{-a''} - \tan^{-1}\frac{\omega' + \omega''}{-(a'+a'')}\right)}{\sqrt{(a'+a'')^2 + (\omega' + \omega'')^2}}\right]\right\} \tag{59}$$

This expression is exact. It gives the integral of the square of the secondary current of the coupled system excited by discharging the primary inductance originally traversed by a current I_1

If, now, we neglect the squares of the damping constants in comparison with the squares of the angular velocities, this equation, by the employment of processes similar to those used in deriving (35), reduces to

$$J = \frac{I_1^2 H^2}{Q_1^2}\left\{\frac{1}{4a'} + \frac{1}{4a''} - \frac{(\omega' - \omega'')^2}{4\omega'\omega''}\sin\left(2\tan^{-1}\frac{\omega' - \omega''}{a'' - a'}\right) - \frac{a'' + a' - \dfrac{a'\omega'' - a''\omega'}{\omega'\omega''(\omega' + \omega'')}(\omega' - \omega'')^2}{(a' + a'')^2 + (\omega' - \omega'')^2}\right. \qquad (60)$$

where

$$H = \frac{MC_2Q_1\omega'^2\omega''^2}{(\omega' + \omega'')\sqrt{(a'' - a')^2 + (\omega' - \omega'')^2}} \qquad (61)$$

Equation (60) gives the integral of the square of the secondary current, in a coupled system, excited by discharging the primary inductance originally traversed by a current I_1, in case the squares of the damping constants are negligible in comparison with the squares of the angular velocities. No other approximation has been made. The Q_1 that occurs in (60) has no meaning, in this case, and is eliminated by the Q_1 occurring in H in (61).

If next we assume the circuits very loosely coupled and assume that they do not depart from synchronism by more than a few per cent., and apply the assumptions and methods employed in deriving (41), we find

$$J = \frac{M^2C_2^2I_1^2\omega^6}{16a_1a_2}\,\frac{a_1 + a_2}{(a_1 + a_2)^2 + (\omega_1 - \omega_2)^2} \qquad (62)$$

where, as before

$$J = \int_0^\infty i_2^2 dt.$$

Equation (62) gives the value of the time integral of the square of the secondary current in a coupled system excited by a discharge of the primary inductance originally traversed by a current I_1. In obtaining this simplified result the squares of the damping constants have been neglected in comparison with the squares of the angular velocities, and the coefficient of coupling has been assumed

to be so small that the damping constants and angular velocities of the coupled systems are the same as these constants for the circuits uncoupled, as expressed in (36). *Also the circuits as supposed to be near enough to synchronism to make* (37) *applicable.*

It is seen that the value of J divided by J_r, which is the value of J at resonance, reduces approximately to the same value as with the [illegible] method of excitation (compare (41)), so that the method of decrement measurement illustrated in Fig. 1 and the text of Art. **140** applies also to the inductance method of excitation.

CHAPTER XI

THEORY OF TWO COUPLED CIRCUITS UNDER THE ACTION OF AN IMPRESSED SINUSOIDAL ELECTROMOTIVE FORCE

In the treatment of two coupled circuits the discussion up to the present has been confined to the free oscillation that takes place when the system is given a charge and is allowed to discharge. It is proposed now to treat the two circuits, when one of them has operating within it, or upon it, a sinusoidal electromotive force.[1]

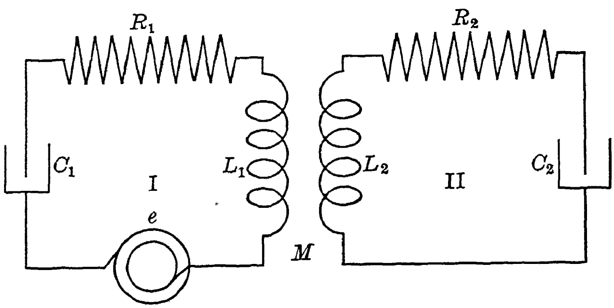

FIG. 1.—Two coupled circuits with impressed e m.f

144. Form of Circuit to Which the Analysis Applies.—The form of circuit to which the analysis is to apply exactly is shown in Fig. 1, where the circuit I contains a condenser, an inductance and a resistance and a source of sinusoidal electromotive force, indicated at e.

Coupled with the circuit I is a secondary circuit II, containing also inductance, resistance, and capacity in series with one another.

The constants of the circuits are L_1, C_1, R_1 for the primary,

[1] This problem without condensers in the circuits was first treated by MAXWELL, *Phil. Trans.*, 155, 1864. With condensers it was treated by BEDELL & CREHORE, *Physical Review*, 1, p. 117 and p. 177, 1893 and 2, p. 442, 1894. See also OBERBECK, *Wied. Ann.*, 55, p. 623, 1895; and PIERCE, *Proc. Am. Acad.*, 46, p. 291, 1911.

and L_2, C_2, R_2 for the secondary circuit. M is the mutual inductance between the two circuits.

145. The Differential Equations.—Let the e.m.f. impressed upon the primary be

$$e = E \cos \omega t = \text{real part of } E\epsilon^{j\omega t} \tag{1}$$

Taking, now, the fall of potential around each of the circuits, and equating it to the impressed e.m.f., we obtain the following differential equations involving the currents in the two circuits:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{\int i_1 dt}{C_1} - M\frac{di_2}{dt} = E\epsilon^{j\omega t} \tag{2}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} + \frac{\int i_2 dt}{C_2} - M\frac{di_1}{dt} = 0 \tag{3}$$

where in equation (2), for simplicity, we have replaced the actual impressed e.m.f., $E \cos \omega t$, which is a real quantity, by a complex quantity

$$E\epsilon^{j\omega t} = E(\cos \omega t + j \sin \omega t) \tag{4}$$

The result is that the solutions that we shall now obtain will give complex quantities for the values of i_1 and i_2. Of these complex values of i_1 and i_2, the real components will be the solution of the given problem with $E \cos \omega t$ as the impressed e.m.f.

146. Nature of the Solution.—The complete solution of the pair of equations (2) (3) is obtained by adding the *particular integral* to the *complementary function.*

The *Complementary Function* in i_1 and i_2 is the general solution of the system (2) (3) with the right-hand side of (2) replaced by zero. This we have obtained in Chapter IX in the form of (21) and (22), Art. **101.** Such a solution for i_1 and i_2 with the arbitrary constants undetermined is to be a part of the solution of our present problem.

The *Particular Integral* of the pair of equations is any pair of values of i_1 and i_2 that will satisfy the simultaneous equations (2) and (3).

147. Determination of the Particular Integral.—It appears that in order to meet the term involving the exponential in $j\omega t$ on the right-hand side of (2), we shall probably need such an exponential in our value of i_1 and i_2. Let us try setting

$$i_1 = A\epsilon^{j\omega t} \tag{5}$$

$$i_2 = B\epsilon^{j\omega t} \tag{6}$$

where ω is specifically the ω of the impressed e.m.f., and is not an unknown quantity to be obtained from the constants of the circuits as was the k in the exponentials in kt employed in Chapters VIII and IX.

To see if the assumed solutions are correct, let us substitute (5) and (6) in (2) and (3), obtaining

$$\epsilon^{jwt}\left[A\left\{R_1 + j\left(L_1\omega - \frac{1}{C_1\omega}\right)\right\} - jM\omega B\right] = E\epsilon^{jwt} \tag{7}$$

$$\epsilon^{jwt}\left[B\left\{R_2 + j\left(L_2\omega - \frac{1}{C_2\omega}\right)\right\} - jM\omega A\right] = 0 \tag{8}$$

In these equations let us designate the *Reactances* of the separate circuits by X_1 and X_2; that is, let

$$X_1 = L_1\omega - \frac{1}{C_1\omega} \tag{9}$$

and

$$X_2 = L_2\omega - \frac{1}{C_2\omega} \tag{10}$$

It is seen that the exponential factors of (7) and (8) divide out; and our assumed solutions prove to be correct provided (7) and (8) are satisfied. These reduce to

$$(R_1 + jX_1)A - jM\omega B = E \tag{11}$$

and

$$(R_2 + jX_2)B - jM\omega A = 0 \tag{12}$$

and completely determine A and B, as we shall soon show.

The complex quantities $R_1 + jX_1$ and $R_2 + jX_2$ that occur in (11) and (12) and, for a given impressed frequency, are constants of the Circuits I and II, and are usually designated by a small z with proper subscript:

$$z_1 = R_1 + jX_1 \tag{13}$$

$$z_2 = R_2 + jX_2 \tag{14}$$

These quantities are called *complex impedances*.

As further abbreviations it is customary to designate the magnitudes of z_1 and z_2 by capital Z_1 and Z_2 defined by

$$Z_1 = \sqrt{R_1^2 + X_1^2} \tag{15}$$

$$Z_2 = \sqrt{R_2^2 + X_2^2} \tag{16}$$

The quantities Z_1 and Z_2 are called *impedances*.

Returning now to the relations (11) and (12) between A and B, these equations in terms of z_1 and z_2 become

$$z_1 A - jM\omega B = E,$$

$$z_2 B - jM\omega A = 0,$$

whence

$$B = \frac{jM\omega A}{z_2} \tag{17}$$

and

$$A = \frac{E}{z_1 + \frac{M^2\omega^2}{z_2}} = \frac{E}{z'_1}(\text{say}), \tag{18}$$

where, as an abbreviation,

$$z'_1 = z_1 + \frac{M^2\omega^2}{z_2} \tag{19}$$

In terms of z'_1, our equations (5) and (6) become

$$i_1 = \frac{E\epsilon^{j\omega t}}{z'_1} \tag{20}$$

$$i_2 = \frac{jM\omega E\epsilon^{j\omega t}}{z_2 z'_1} \tag{21}$$

Since in (20) the quantity z'_1 occurs as a divisor of the complex e.m.f. to give the complex current, we may call z'_1 the *apparent complex impedance of the primary circuit.* We may analyze z'_1 into its real and imaginary parts by replacing z_1 and z_2 by their values (13) and (14). Then (19) becomes

$$z'_1 = R_1 + jX_1 + \frac{M^2\omega^2}{R_2 + jX_2}.$$

Rationalizing the second term, we obtain

$$z'_1 = R_1 + \frac{M^2\omega^2}{Z_2^2} R_2 + j\left\{ X_1 - \frac{M^2\omega^2}{Z_2^2} X_2 \right\} \tag{22}$$

$$= R'_1 + jX'_1 \text{ (say)}, \tag{23}$$

where

$$R'_1 = R_1 + \frac{M^2\omega^2}{Z_2^2} R_2 \tag{24}$$

and

$$X'_1 = X_1 - \frac{M^2\omega^2}{Z_2^2} X_2 \tag{25}$$

If we should replace z'_1, of equation (20) by its value as given

in (23), we should see that the current for the primary circuit would be the same as it would be if the secondary circuit were not present, provided the primary resistance were changed to R'_1 and the primary reactance to X'_1. These quantities R'_1 and X'_1 are called respectively the *apparent resistance* and *apparent reactance* of the primary circuit.

It may be noted that the apparent resistance is greater than the true resistance; but, since X_2 may be positive, negative or zero depending on the relative values of $L_2\omega$ and $C_2\omega$, the apparent reactance may be greater than, equal to, or less than, the true reactance of the primary circuit alone.

If now we introduce a quantity called *apparent impedance*, indicated by Z'_1, and defined by

$$Z'_1 = \sqrt{R'^2_1 + X'^2_1} \tag{26}$$

and also introduce the abbreviation

$$\varphi'_1 = \tan^{-1}\frac{X'_1}{R'_1}, \tag{27}$$

we may write (23) in the form

$$z'_1 = Z'_1\epsilon^{j\varphi'_1} \tag{28}$$

We are going to use this equation in determining the real component of i_1.

In like manner, for the determination of i_2, we may employ

$$z_2 = Z_2\epsilon^{j\tan^{-1}\frac{X_2}{R_2}} \tag{29}$$

and

$$j = \epsilon^{j\pi/2} \tag{30}$$

Substituting (28), (29) and (30) into (20) and (21), and taking the real part of the results, we obtain

$$i_1 = \frac{E}{Z'_1}\cos(\omega t - \varphi'_1) \tag{31}$$

$$i_2 = \frac{M\omega E}{Z_2Z'_1}\cos\left(\omega t - \varphi'_1 + \pi/2 - \tan^{-1}\frac{X_2}{R_2}\right) \tag{32}$$

Equations (31) *and* (32) *are the required particular integrals of the differential equations* (2) *and* (3), *with, however, the exponential e.m.f. of* (2) *replaced by E cos ωt. All of the quantities entering into these expressions are known in terms of the constants of the circuits and the amplitude and angular velocity of the impressed e.m.f.* Z'_1 *and* φ'_1 *are defined respectively by* (26) *and* (27).

148. The Complete Solution and the Steady State Solution.—As pointed out above, the complete solution of the given differential equations is made up of the particular solutions (31) and (32) plus the values of i_1 and i_2 respectively given by equations (21) and (22) of Chapter IX. The latter are the values of the currents for a free oscillation of the circuits. These currents are doubly periodic in general with angular velocities ω' and ω'' and damping constants a' and a'' determined by the constants of the circuits. Superposed on this doubly periodic free oscillation, are the current values given as our particular solutions (31) and (32). These particular solutions have the frequency of the impressed e.m.f., and are hence called the *forced solutions*.

After a sufficient lapse of time the free solution terms, which have exponential damping, subside and leave only the terms given in (31) and (32). These values of i_1 and i_2 given by (31) and (32) constitute *the steady state values of the currents.*

We may note then that the steady-state currents have the frequency of the impressed e.m.f., and are completely given by (31) *and* (32). *Whenever these equations are used as the complete values of the currents, we must make sure that a sufficient time has elapsed after the application of the e.m.f. to permit the subsidence of the transient terms of the form of those obtained in Chapter IX as the free oscillation currents of the system.*

PARTIAL RESONANCE. MAXIMUM AMPLITUDE OF SECONDARY CURRENT OBTAINED BY ADJUSTMENT OF A SINGLE VARIABLE

149. Attention to Secondary Current Amplitude.—We shall for the present confine our attention to the amplitude of the current in the Circuit II, which may be called the secondary circuit, since the e.m.f. is applied to the other circuit, Circuit I.

Both in the case of the sending station and the receiving station this secondary current is important; for in the case of a sending station the e.m.f. is applied usually to a closed circuit coupled with an antenna circuit, so that *the secondary circuit would be the antenna circuit at the sending station,* and we are interested in knowing the current in the antenna. At the receiving station the e.m.f. may be regarded as impressed on the antenna from a distant station, while coupled with the receiving antenna is usually a closed circuit actuating the detector. *This closed circuit would, therefore, be a secondary*

circuit with reference to the receiving antenna, and we are interested in knowing the current received in this secondary circuit.

We shall here limit the investigation to conditions for producing a maximum amplitude of current, in a steady state, in the secondary circuit, Circuit II, under the action of a sinusoidal e.m.f. in Circuit I.

150. Definitions of Partial Resonance Relations S and P.—When any single element of the system is adjusted to produce a maximum secondary current amplitude, while all the other elements are kept constant, we shall designate the condition as one of *Partial Resonance* and shall describe the adjusted member as satisfying a *Partial Resonance Relation.*

Two partial resonance relations will now be derived, and will be designated S and P, where S means that the secondary is adjustable. P means that the primary is adjustable.

Partial Resonance Relation P will be used to describe the adjustment of the primary reactance X_1 that will give a maximum amplitude of secondary current, when all the other elements of the system are kept constant. The result will appear as an equation for the determination of X_1.

Partial Resonance Relation S will designate the adjustment of the Secondary reactance X_2 that will give a maximum of amplitude of secondary current, when all the other elements of the system are kept constant.

It is evident that these two partial resonance relations are determined mathematically by setting severally equal to zero the partial derivatives of I_2 with respect to X_1 and X_2.

We shall now proceed to determine these partial resonance relations.

151. Determination of Partial Resonance Relation S.—Denoting the amplitude of current in the secondary circuit by I_2, we have from (32)

$$I_2 = M\omega E/Z_2Z'_1 \tag{33}$$

where

$$Z_2^2 = R_2^2 + X_2^2$$

and

$$Z'^2_1 = \left(R_1 + \frac{M^2\omega^2}{Z_2^2}R_2\right)^2 + \left(X_1 - \frac{M^2\omega^2}{Z_2^2}X_2\right)^2 \tag{34}$$

Since in (33) M, ω, and E are to be considered constant, and since Z_2 and Z'_1 are both positive, we may obtain a maximum value of I_2 by determining the condition for a minimum value

of the square of the denominator of (33). With X_2 as the variable, this is done by setting equal to zero the derivative of the square of the denominator of (33) with respect to X_2.

That is

$$0 = \frac{\partial}{\partial X_2}\left\{Z_2^2 Z'^2_1\right\}.$$

Expanding Z'_1 by (34) and multiplying by Z_2^2, we obtain

$$Z_2^2 Z'^2_1 = Z_2^2 Z_1^2 + M^4\omega^4 + 2M^2\omega^2(R_1R_2 - X_1X_2) \quad (35)$$

Performing the operation indicated by the equation preceding (35), we obtain

$$0 = 2X_2Z_1^2 - 2M^2\omega^2X_1;$$

whence our required condition for a maximum amplitude of secondary current becomes

$$\frac{X_2}{X_1} = \frac{M^2\omega^2}{Z_1^2} \text{ (Partial Resonance Relation } S\text{).} \quad (36)$$

This equation (36) *gives the value that* X_2 *must have in order to give a maximum current in the secondary circuit when all the quantities except* X_2 *are kept constant. The relation* (36) *will be called* **Partial Resonance Relation S.**

152. Partial Resonance Relation P.—Let us now return to the general expression (33) for I_2, and suppose that, with any arbitrary fixed values of R_1, R_2, M, ω, and X_2, it be required to determine what adjustment of the Primary Reactance X_1 is necessary in order to make the secondary current a maximum. That is, instead of adjusting the secondary reactance X_2 we are going to adjust the primary reactance X_1 to give the maximum current amplitude in the secondary circuit.

The result in this case can be obtained by inspection, for Z_2 does not involve X_1. In the denominator of (33) only Z'_1 involves X_1, and we must choose X_1 to make Z'_1 a minimum. By (34) it is seen that this is attained by making the expression in the last parenthesis in (34) zero; that is

$$\frac{X_1}{X_2} = \frac{M^2\omega^2}{Z_2^2} \text{ (Partial Resonance Relation } P\text{).} \quad (37)$$

Equation (37) *gives* **Partial Resonance Relation P,** *which determines the value that* X_1 *must have in order for the secondary current amplitude to be a maxium for the given fixed values of* X_2, $M^2\omega^2$ *and* Z_2.

153. Note Regarding Effect of Resistances on Partial Resonance Relations P and S.—In equation (36), Z_1 contains R_1 as one of its terms, while in (37) Z_2 contains R_2 as one of its terms. The resistances do not enter otherwise in these two expressions.

It is to be noted then that the resistance of the secondary circuit has no effect in determining the adjustment that must be given to the secondary reactance to make the secondary current a maximum; and the resistance of the primary circuit has no effect in determining the adjustment that must be given to the primary reactance to make the secondary current a maximum.

154. Secondary Current Under Partial Resonance Relation S.—Let us obtain next the current amplitude in the secondary circuit when the secondary reactance is adjusted to the partial resonance relation S, as given in (36).

To do this let us substitute the value of X_2 from (36) into (35) and extract the square root of (35) to get the denominator of (33). In making this substitution Z_2^2 of the right-hand side of (35) must be decomposed into $R_2^2 + X_2^2$, so that the X_2^2 may be replaced. When we have made this substitution we shall have imposed upon I_2 the condition (resonance relation S) for a maximum; therefore we shall write the resulting value of I_2 as $[I_{2\max}]_S$ We obtain

$$[I_{2\max}]_S = \frac{M\omega E}{\sqrt{Z_1^2\left(R_2^2 + \frac{M^4\omega^4X_1^2}{Z_1^4}\right) + M^4\omega^4 + 2M^2\omega^2\left(R_1R_2 - \frac{M^2\omega^2X_1^2}{Z_1^2}\right)}}$$

which reduces to

$$[I_{2\max.}]_S = \frac{M\omega E}{R_2Z_1 + \frac{M^2\omega^2R_1}{Z_1}} \tag{38}$$

Equation (38) gives the current amplitude in the secondary circuit, when for fixed values of the other constants of the circuits, X_2 is set at the value to give a maximum secondary current amplitude. Expressed otherwise, (38) gives the amplitude of secondary current under partial resonance relation S.

155. Secondary Current Under Partial Resonance Relation P. In like manner, if we substitute (37) into (33) and designate the resulting value of I_2 by $[I_{2\max}]_P$, we obtain

$$[I_{2\max}]_P = \frac{M\omega E}{R_1Z_2 + \frac{M^2\omega^2R_2}{Z_2}} \tag{39}$$

Equation (39) gives the amplitude of secondary current under Partial Resonance Relation P; that is, under the condition that for fixed values of the other constants of the circuits, X_1 is set at the value to give maximum amplitude of secondary current.

II. THE OPTIMUM RESONANCE RELATION

156. The Optimum Resonance Relation.—For given values of certain constants of the coupled system we have found two different adjustments, one of the primary reactance, and the other of the secondary reactance, that would give a maximum amplitude of secondary current. In order to get the biggest possible current in the secondary circuit, it is apparent that we should, if possible, satisfy the Partial Resonance Relation S and the Partial Resonance Relation P both at the same time.

It is somewhat more instructive to proceed by another method as follows:

Equation (36) tells us what value we must give to the reactance X_2, of Circuit II, for a given X_1, Z_1, E, M, and ω, in order to obtain a maximum amplitude of current in Circuit II.

If now we take a different set of values of these constants X_1, Z_1, we shall require a different value of X_2, and shall get a different maximum value of secondary current. We may now ask ourselves which of these several combinations of adjustments will give a maximum of the maxima of secondary current amplitude.

To determine this let us suppose that X_2 is always automatically given the value that satisfies resonance relation S, so that (38) is kept satisfied, even as we vary X_1, and let us determine the value of X_1 that under this condition will give a maximum of $[I_{2\max}]_S$.

This is attained mathematically by setting equal to zero the derivative of the denominator of (38); that is

$$
\begin{aligned}
0 &= \frac{\partial}{\partial X_1}\left[R_2 Z_1 + \frac{M^2\omega^2 R_1}{Z_1}\right] \\
&= \left[R_2 - \frac{M^2\omega^2 R_1}{Z_1^2}\right]\frac{\partial Z_1}{\partial X_1}.
\end{aligned} \tag{40}
$$

Now by definition

$$Z_1 = \sqrt{R_1^2 + X_1^2},$$

so

$$\frac{\partial Z_1}{\partial X_1} = X_1/\sqrt{R_1^2 + X_1^2} = X_1/Z_1.$$

This put into the second form of (40), gives, after multiplication by Z_1,

$$0 = X_1(R_2 - M^2\omega^2 R_1/Z_1^2).$$

From this it follows that one or the other of the following equations is the condition of the required maximum of $[I_{2\max}]_S$; to wit:

$$\text{Either} \qquad X_1 = 0 \tag{I}$$

$$\text{or} \qquad \frac{R_2}{R_1} = \frac{M^2\omega^2}{Z_1^2} \tag{II}$$

We are now to decide which of these two conditions, (I) or (II), is correct for determining the required maximum of $[I_{2\max}]_S$. Let us first replace Z_1^2 by its value $X_1^2 + R_1^2$, which, substituted into (II), gives

$$X_1^2 = \frac{R_1}{R_2}(M^2\omega^2 - R_1R_2) \qquad \text{(II')} \qquad (41)$$

Equation (II′) is equivalent to (II).

Let us examine two cases.

Case I. Let

$$M^2\omega^2 < R_1R_2.$$

In this case the proper resonance relation is (I), for if $M^2\omega^2$ is less than R_1R_2, Condition (II′) makes X_1 imaginary and is therefore unattainable.

Case II. Let

$$M^2\omega^2 > R_1R_2.$$

By substitution of Conditions (I) and (II) severally into (38) we find that Condition (I) reduces the denominator of (38) to

$$R_1R_2 + M^2\omega^2 = A\text{(say)};$$

while Condition (II) reduces this denominator to

$$2M\omega\sqrt{R_1R_2} = B\text{(say)}.$$

Now B is seen to be less than A, because twice the product of any two real quantities is less than the sum of their squares. Hence in this case Condition (II) gives a larger amplitude of secondary current than does (I).

If $M^2\omega^2 = R_1R_2$, Conditions (I) and (II) reduce to the same condition as may be seen by comparing (II′) with (I).

It thus appears that under the limitations of Case I, Condition

(I) gives the largest attainable secondary current; and under the limitations of Case II, condition (II) gives the largest attainable secondary current; and, if $M^2\omega^2 = R_1R_2$, Conditions (I) and (II) are both appropriate for giving the largest possible current, in the secondary circuit.

These results have been attained by supposing that, while seeking the optimum condition, we have kept (36) always satisfied; so (36) must be fulfilled simultaneously with (I) when (I) is optimum and simultaneous with (II) when (II) is optimum.

Combining (36) with (I) and (II) in the two cases we have respectively the results following.

$$\text{If} \qquad M^2\omega^2 < R_1R_2 \tag{42}$$

$$\text{then} \qquad X_1 = 0, \quad \text{and} \quad X_2 = 0 \tag{43}$$

gives the largest attainable amplitude of secondary current. We shall call this system of equations **the optimum resonance relation at deficient coupling.**

On the other hand, if

$$M^2\omega^2 > R_1R_2 \tag{44}$$

the combination of (II) with (36) gives

$$\frac{X_2}{X_1} = \frac{R_2}{R_1} = \frac{M^2\omega^2}{Z_1^2} \tag{45}$$

as the condition for the largest attainable amplitude of secondary current. We shall call the system of equations (44) and (45) **the optimum resonance relation at sufficient coupling.**

In the interest of completion of nomenclature, if

$$M^2\omega^2 = R_1R_2 \tag{46}$$

we shall call the coupling **critical coupling.** Either (43) or (45) is **the optimum resonance relation at critical coupling,** since both reduce to the form (43) as may be seen from (41).

If (42) *is fulfilled,* (43) *is the condition for maximum amplitude of secondary current. If, on the other hand* (44) *is fulfilled,* (45) *gives this condition. If* (46) *is fulfilled,* (45) *and* (43) *reduce to the same value.*

157. Value of Max. Max. Secondary Current Amplitude at Deficient Coupling.—The case of deficient coupling is the case in which

$$M^2\omega^2 < R_1R_2. \tag{47}$$

Then the appropriate settings of the two circuits for the greatest possible amplitude of secondary current is the adjustment that makes

$$X_1 = 0 = X_2; \tag{48}$$

that is, each circuit is separately adjusted so as to make its undamped period equal to the period of the impressed e.m.f. From (38) the current obtainable under these conditions is

$$I_{2\,\max\,\max} = \frac{M\omega E}{R_1R_2 + M^2\omega^2}. \tag{49}$$

If the circuits are so loosely coupled that $M^2\omega^2 < R_1R_2$, then for a max. max. secondary current, the circuits should be tuned to satisfy (48), *and the current obtained at this adjustment is given by* (49).

The current is seen to decrease with decreasing M, for if we differentiate (49) with respect to M we obtain a negative quantity for all values of $M^2\omega^2$ less than R_1R_2.

158. Value of Max. Max. Secondary Current at Sufficient Coupling.—In this case

$$M^2\omega^2 > R_1R_2. \tag{50}$$

The appropriate setting of the two circuits for the greatest possible secondary current in this case is given by equations (45) which are here rewritten

$$\frac{X_2}{X_1} = \frac{R_2}{R_1} = \frac{M^2\omega^2}{Z_1^2} \tag{51}$$

As an alternative expression, it has been seen that the Condition (II′) of equation (41) was equivalent to the Condition (II), preceding (41), which combined with the first part of (40) gives

$$\left.\begin{aligned} X_1 &= \pm\sqrt{\frac{R_1}{R_2}(M^2\omega^2 - R_1R_2)} \\ \text{and}\qquad & \\ X_2 &= \pm\sqrt{\frac{R_2}{R_1}(M^2\omega^2 - R_1R_2)}. \end{aligned}\right\} \tag{52}$$

Equations (52) *are together equivalent to* (51) *provided both radicals in* (52) *are given the same sign.*

Now from the second part of (51) we obtain

$$Z_1 = M\omega\sqrt{\frac{R_1}{R_2}}.$$

If we substitute this quantity into (38) and call the resultant current amplitude $I_{2\ \max\ \max}$, we have

$$I_{2\ \max\ \max.} = \frac{E}{2\sqrt{R_1R_2}}. \tag{53}$$

If the circuits are so closely coupled as to satisfy the condition for **sufficient coupling** *as defined by* (50), *then in order to obtain a max. max. secondary current, the circuits should be tuned to satisfy conditions* (51), *or the equivalent conditions* (52), *and the current obtained at this adjustment is given by* (53).

It is seen that in the case of sufficient coupling (that is, when $M^2\omega^2 > R_1R_2$) the value of the secondary current obtained *is independent of the mutual inductance.*

159. Optimum Resonance Relation Equivalent to Fulfillment of Partial Resonance Relations S and P Simultaneously.—Before passing to a further consideration of max. max. current amplitudes it is interesting to note that the simultaneous fulfillment of Partial Resonance Relation S and Partial Resonance Relation P results in the Optimum Resonance Relation.

The Partial Resonance Relation S given by (36) is

$$\frac{X_2}{X_1} = \frac{M^2\omega^2}{Z_1^2}, \tag{S}$$

while the Partial Resonance P given by (37) is

$$\frac{X_1}{X_2} = \frac{M^2\omega^2}{Z_2^2}. \tag{P}$$

Taking the product and then the quotient of these two equations, we obtain

$$Z_1Z_2 = M^2\omega^2 \tag{54}$$

and

$$\frac{X_2^2}{X_1^2} = \frac{Z_2^2}{Z_1^2} = \frac{X_2^2 + R_2^2}{X_1^2 + R_1^2} = \frac{R_2^2}{R_1^2} \tag{55}$$

The last step in (55) is taken by the law of *division* in the theory of ratio and proportion.

Taking the square root of the first and last members of (55) and combining with (S) we have the optimum resonance relation (51), which is the case of sufficient coupling.

Note, however, (S) and (P) are attainable simultaneously only provided (54) is attainable, but since by definitions of Z_1 and Z_2,

$$Z_1Z_2 \geqq R_1R_2;$$

hence, by (54) (S) and (P) are simultaneously attainable only provided

$$M^2\omega^2 \gtreqless R_1R_2.$$

This is not quite correct, because there is another way of satisfying (S) and (P) simultaneously without leading to (54), and that is by making

$$X_1 = 0 \text{ and } X_2 = 0 \tag{56}$$

so that alternative to the optimum resonance relation (55) we have (56) as a possible optimum resonance relation. By work done above, it was shown that (56) is the actual optimum resonance relations, provided

$$M^2\omega^2 < R_1R_2.$$

We have thus shown that the Optimum Resonance Relation is Equivalent to the requirement that the Partial Resonance Relations P and S be fulfilled simultaneously.

Instantaneous Value of Secondary Current and of Primary Current at Optimum Resonance. Sufficient Coupling.—Under the conditions for optimum resonance for sufficient coupling the apparent resistance and the apparent reactance of the primary circuit, as given in (24) and (25), reduce to

$$R'_1 = 2R_1, \quad X'_1 = 0 \tag{57}$$

whence the angle φ'_1 as defined in (27) reduces to

$$\varphi'_1 = 0 \tag{58}$$

The instantaneous current i_1, as given by (31), under these conditions reduces to

$$i_{1\,\max\,\max} = \frac{E \cos \omega t}{2R_1} \tag{59}$$

This equation gives the value of the instantaneous current in the primary circuit at optimum resonance and sufficient coupling. In this equation $E \cos \omega t$ is the impressed e.m.f., and the result is for the steady state.

Next, to determine the secondary instantaneous current, let us take (32), replace its amplitude by (53), and also make $\theta'_1 = 0$, as in (58), obtaining

$$i_{2\,\max\,\max.} = \frac{E \cos\left(\omega t + \pi/2 - \tan^{-1}\dfrac{X_1}{R_1}\right)}{2\sqrt{R_1R_2}} \tag{60}$$

This equation gives the value of the instantaneous current in the secondary circuit at optimum resonance with sufficient coupling, in a steady state, under the action of an e.m.f. E cos ωt impressed upon the primary.

POWER EXPENDITURE IN THE COUPLED CIRCUITS

160. Power Expended in the Primary and Secondary Circuits in the Coupled System at Optimum Resonance for Sufficient Coupling.—If we multiply the instantaneous e.m.f. E cos ωt by the instantaneous primary current (59) at optimum resonance (sufficient coupling), we obtain for the instantaneous power p_1 supplied to the primary circuit

$$p_1 = \frac{E^2 \cos^2 \omega t}{2R_1} \tag{61}$$

If we take the time average of this power, over an integral number of half-periods, or over a time that is long in comparison with a half-period, and indicate the average so obtained by P_1, the average of the numerator becomes $E^2/2$. This value is the mean square of e, which mean square we may indicate by $\overline{E^2}$, and obtain

$$P_1 = \overline{E^2}/2R_1 \tag{62}$$

This is the average power-input into the system of circuits, at optimum resonance with sufficient coupling.

Next, let us examine the power converted into heat or radiated as electric waves from the primary circuit. This is the square of the current times the resistance of the circuit. If we call this power $[p_1]_R$, we have

$$[p_1]_R = i_1{}^2 R_1 = \frac{E^2 \cos^2 \omega t}{4R_1} \tag{63}$$

of which the average, indicated by replacing p by capital P, is

$$[P_1]_R = \overline{E^2}/4R_1 \tag{64}$$

Equations (63) *and* (64) *give respectively the instantaneous power and the average power converted into heat in the primary circuit or radiated from it as electric waves, at optimum resonance with sufficient coupling.*

The difference between the power-input and the power converted in the primary circuit is the power communicated to the

secondary circuit. By taking (63) from (61) and (64) from (62), this is seen to be

$$p_{12} = \frac{E^2 \cos^2 \omega t}{4R_1} \tag{65}$$

and

$$P_{12} = \overline{E^2}/4R_1 \tag{66}$$

Equations (65) *and* (66) *give respectively the instantaneous power and the average power communicated to the secondary circuit at optimum resonance with sufficient coupling. These values are seen to be the same as the corresponding quantities converted in the primary into heat or radiated from it.*

Let us now as an independent operation calculate the power consumed in the resistance of the secondary circuit. This is obtained by multiplying the square of the instantaneous secondary current (60) by the secondary resistance R_2, and gives

$$p_2 = \frac{E^2 \sin^2[\omega t - \tan^{-1}(X_1/R_1)]}{4R_1} \tag{67}$$

of which the time average is

$$P_2 = \overline{E^2}\,/4R_1 \tag{68}$$

These equations (67) *and* (68) *give the instantaneous power and the average power converted into heat or radiation in the secondary circuit. It is seen that the average value is the same as the average value of power communicated to the secondary from the primary, and the same as the average power consumed in the primary.*

A comparison of the instantaneous values (67) and (65) shows that the conversion into heat is not in phase with the transfer from the primary to the secondary. This is not surprising for the power, for a part of the time, is stored in the condenser and inductance of the secondary circuit.

As a general conclusion from this investigation into power the important result is obtained that, with $M^2\omega^2$ greater than R_1R_2, if we adjust the two circuits to such values as to give a max. max. of *secondary current, then one-half of all the power communicated to the system through the impressed e.m.f. is dissipated in the primary circuit and one-half is dissipated in the secondary circuit.*

This adjustment is, therefore, not a very efficient one, in general, for communicating power to a coupled system and dissipating it in a secondary load.

If on the other hand, our problem is the reception of electric waves from a distant station by means of a coupled system of

circuits and the affecting of an instrument in the secondary circuit, which instrument responds more actively the larger the secondary current, this adjustment though not efficient may give the maximum of response in the receiving instrument. It is to be noted, however, that we have assumed a constant amplitude of impressed e.m.f., and if the radiation from the receiving antenna affects the resultant impressed e.m.f., a proper correction has to be applied.

We shall next discuss the conditions for maximum efficiency of transfer of power to the secondary circuit through the coupled system.

161. Condition for the Transfer of Power to the Secondary Circuit with Maximum Efficiency.—We must now go back to our original current equations (31) and (32), unmodified by the introduction of any resonance relations, and form the expressions for the average power expended in the secondary resistance and the average power expended in the primary resistance.

This is done by taking the square of the respective currents and multiplying by the respective resistances and averaging as to time. If we merely write the ratio of these average power values, we obtain

$$\frac{P_2}{P_1} = \frac{M^2\omega^2 R_2}{Z_2^2 R_1} \tag{69}$$

It is seen that, for a fixed value of M, ω, R_2, and R_1, this ratio of the average power expended in the secondary to the average power expended in the primary is a maximum when X_2, comprised in Z_2, is zero. That is,

$$X_2 = 0 \tag{70}$$

Equation (70) is the condition for a maximum efficiency of the transfer of power to the secondary circuit.

Putting (70) into (69), it is seen that at maximum efficiency

$$\frac{P_2}{P_1} = \frac{M^2\omega^2}{R_1 R_2} \tag{71}$$

To obtain from this expression the efficiency at maximum efficiency it is only necessary to form from (71) the ratio $P_2/(P_1 + P_2)$. This is done by taking the reciprocal of (71), adding unity to both sides, and again taking the reciprocal. This gives

$$\text{Eff.}_{\text{max.}} = \frac{M^2\omega^2}{M^2\omega^2 + R_1 R_2} \tag{72}$$

Equation (72) gives the efficiency of the transfer of power from the impressed e.m.f. to the secondary circuit when the secondary circuit is adjusted for the maximum efficiency of such transfer. The efficiency of the transfer is independent of the primary adjustment.

162. Condition for the Transfer of Maximum Power to the Secondary Circuit, The Transfer Being Effected at Maximum Efficiency.—If we want to get the maximum transfer of power to the secondary circuit at maximum efficiency, we need merely put the condition for the maximum efficiency of transfer (namely, $X_2 = 0$) into the amplitude equation (33) for secondary current and then adjust X_1 to make the square of this amplitude a maximum

Putting $X_2 = 0$ into (33) we obtain

$$I_2 = \frac{M\omega E}{R_2\sqrt{(R_1 + \frac{M^2\omega^2}{R_2})^2 + X_1^2}}. \tag{73}$$

It is seen by inspection that to make this a maximum, we require X_1 to be zero.

We have then

$$X_1 = 0 = X_2, \text{ or } L_1C_1 = L_2C_2 = 1/\omega^2. \tag{74}$$

Equations (74) are the conditions for a maximum transfer of power at maximum efficiency from the e.m.f. to the secondary circuit. In this equation ω is the angular velocity of impressed e.m.f.

163. Comparison of Secondary Current at Maximum Power and Maximum Efficiency with the Secondary Current at the Optimum Resonance Relation.—The amplitude of the secondary current at maximum secondary power and at maximum efficiency of transfer of power is obtained by inserting (74) into (73). This gives

$$I_{2\,\max\,\text{eff}} = \frac{M\omega E}{M^2\omega^2 + R_1R_2}. \tag{75}$$

This is the secondary current at maximum secondary power transferred at maximum efficiency from the source to the secondary.

Let us compare with this the secondary current at optimum resonance, with coupling sufficient, which by (53) is

$$I_{2\,\max\,\max} = \frac{E}{2\sqrt{R_1R_2}}.$$

The combination of this equation with (75) gives

$$\frac{I_{2\ \max\ \mathrm{eff}}}{I_{2\ \max\ \max}} = \frac{2\sqrt{R_1R_2/M^2\omega^2}}{1 + R_1R_2/M^2\omega^2} \tag{76}$$

Table I contains calculated values of this ratio for different values of $M^2\omega^2/R_1R_2$.

Table I.—Comparison of Secondary Current for Two Sets of Conditions

$\frac{M^2\omega^2}{R_1R_2}$	Eff max	$\frac{I_2 \text{ max eff}}{I_2 \text{ max. max}}$
1	0.50	1.00
2	0 66	0.93
3	0.75	0.87
4	0.80	0 80
5	0.83	0.74
6	0 86	0 70
7	0.87	0.66
8	0 88	0.62
9	0 90	0 60
∞	1 00	0.00

In the first column of Table I are arbitrary values of the ratio of $M^2\omega^2$ to R_1R_2. Consistent with these ratios, the second column gives the maximum attainable efficiency of the transfer of power to the secondary circuit from the source of e.m.f. This efficiency increases as the ratio in the first column increases. In the third column is the ratio of the amplitude of the secondary current obtainable at maximum efficiency to the amplitude attainable at the adjustment for maximum secondary current. It is seen that at 50 per cent. efficiency this ratio is unity, while with increasing efficiency this ratio decreases toward zero.

CHAPTER XII

RESONANCE RELATIONS IN RADIOTELEGRAPHIC RECEIVING STATIONS UNDER THE ACTION OF PERSISTENT INCIDENT WAVES

164. Use of Persistent Waves.—Persistent, or sustained, waves have recently come into extensive use in radiotelegraphy and radiotelephony. With these persistent waves, which are emitted by the sending station while the sending key is depressed, tens of thousands of oscillations may arrive at the receiving station even during the production of a single dot of the telegraphic code. This permits the establishment of practically

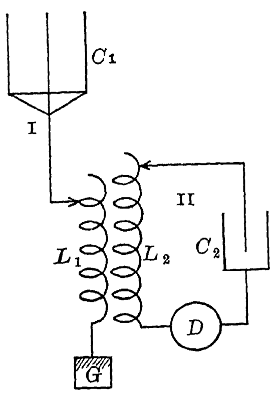

FIG. 1.

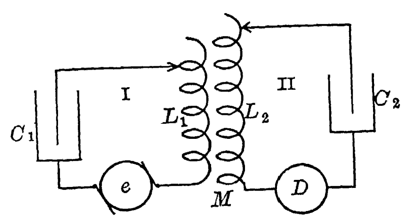

FIG. 2.

FIG. 1—Inductively coupled radiotelegraphic receiving station with detector D in series in a secondary circuit.

FIG. 2.—Closed system approximately equivalent to Fig. 1.

a steady state at the receiving station, so that the mathematical deductions of the preceding chapter may be applied directly to the radiotelegraphic circuits [1]

165. In Respect to Resonance the Antenna Circuit is Approximately Equivalent to a Closed Circuit Consisting of a Localized Inductance, Capacity and Resistance.—With a receiving station of the type shown diagrammatically in Fig. 1,

[1] This chapter is adapted from PIERCE, "Theory of Coupled Circuits, Under the Action of an Impressed Electromotive Force with Applications to Radiotelegraphy," *Proc. Am. Acad*, 46, p. 293, 1911.

certain theory and experiments, not here presented, show that in respect to resonance relations, the system is substantially equivalent to the system of Fig. 2, with antenna replaced by a suitable localized capacity, inductance and resistance.

The e.m.f. impressed upon the antenna by the incoming waves may be simulated by a source e of e.m.f., Fig. 2, in series in the primary circuit.

In the form of receiving circuit illustrated in Fig. 1, the detector D is in series in the secondary circuit, Circuit II, and this whole system goes over into the system of Fig. 2.

In case the detector is of high resistance, it may be advantageous to take it out of Circuit II, and place it along with a condenser C_3 on a branch in shunt to C_2. This arrangement is shown in Fig. 1 of Chapter XV and is there treated. At present we shall suppose the receiving station to be of the type of Fig. 1, and to be equivalent to the simplified system given in Fig. 2.

All that we have developed in the preceding chapter we shall now assume to apply approximately to Fig. 1, and shall describe our results in terms of the radiotelegraphic circuits of this Fig. 1. It is to be borne in mind that what we shall say applies with greater accuracy to the simplified circuits of Fig. 2.

I. PARTIAL RESONANCE RELATIONS S AND P

166. Transformation of Partial Resonance Relations S and P. If Circuit I and the mutual inductance of the system is kept constant and the reactance X_2 of the secondary circuit is used in tuning to obtain a maximum of amplitude of current in Circuit II, the setting required is said to satisfy *Partial Resonance Relation S*. This relation is given in the previous chapter by equation (36), which is here rewritten

$$X_2 = \frac{M^2\omega^2}{Z_1^2}X_1 \text{ (Partial Resonance Relation } S). \qquad (1)$$

On the other hand, if X_1 is used as the adjustable member while all of the other members of the system of circuits are kept constant, the condition for a maximum amplitude of secondary current (in Circuit II) has been called in the previous chapter *Partial Resonance Relations P*. The equation for

this resonance relation is given as (37) of the preceding chapter, and is here rewritten

$$X_1 = \frac{M^2\omega^2}{Z_2^2}X_2 \text{ (Partial Resonance Relation } P). \qquad (2)$$

It is proposed now to transform these two resonance relations by replacing X_1, X_2, Z_1 and Z_2 by their customary values. given respectively in (9), (10), (15) and (16) of Chapter XI, This operation gives

$$L_2\omega - \frac{1}{C_2\omega} = \frac{M^2\omega^2\left(L_1\omega - \dfrac{1}{C_1\omega}\right)}{\left(L_1\omega - \dfrac{1}{C_1\omega}\right)^2 + R_1^2} \text{ (Resonance Relation } S) \quad (3)$$

and

$$L_1\omega - \frac{1}{C_1\omega} = \frac{M^2\omega^2\left(L_2\omega - \dfrac{1}{C_2\omega}\right)}{\left(L_2\omega - \dfrac{1}{C_2\omega}\right)^2 + R_2^2} \text{ (Resonance Relation } P). \quad (4)$$

We shall now change the form of these equations so that the result is expressed in terms of angular velocities, decrements, and the coefficient of coupling. For this purpose, let

$$\Omega_1^2 = 1/L_1C_1, \qquad \Omega_2^2 = 1/L_2C_2, \quad \tau^2 = M^2/L_1L_2 \qquad (5)$$

and let

$$\eta_1 = R_1/L_1\omega = R_1T/2\pi L_1 = \delta_1/\pi \qquad (6)$$

$$\eta_2 = R_2/L_2\omega = R_2T/2\pi L_2 = \delta_2/\pi \qquad (7)$$

The quantities Ω_1 and Ω_2 as defined by (5) are quantities that have been extensively used in Chapters VI, IX, and X and have been designated *Undamped Angular Velocities.* The quantity τ, called *Coefficient of Coupling,* has also been extensively used in the previous chapters.

The quantities η_1 and η_2, defined by (6) and (7), are new, and are seen to be respectively $1/\pi$ times the logarithmic decrements of the two circuits *per cycle of impressed e.m.f.*

Introducing these various abbreviations into (3) and (4) we may write these equations, after a transposition of terms, in the forms

$$\left(1 - \frac{\Omega_1^2}{\omega^2}\right)\left(1 - \frac{\Omega_2^2}{\omega^2}\right) = \tau^2 - \eta_1^2\left\{\frac{1 - \Omega_2^2/\omega^2}{1 - \Omega_1^2/\omega^2}\right\} \qquad (8)$$

(Partial Resonance Relation S)

and

$$\left(1 - \frac{\Omega_2^2}{\omega^2}\right)\left(1 - \frac{\Omega_1^2}{\omega^2}\right) = \tau^2 - \eta_2^2\left\{\frac{1 - \Omega_1^2/\omega^2}{1 - \Omega_2^2/\omega^2}\right\} \tag{9}$$

(Partial Resonance Relation P)

For any given fixed values of the other quantities that occur in these equations, and for fixed amplitude of the impressed e.m.f., equation (8) *gives the value that the ratio* Ω_2/ω *must have in order to produce a maximum of amplitude of secondary current in a steady state.*

Likewise, for the other quantities fixed, equation (9) *gives the value that the ratio* Ω_1/ω *must have in order to produce a maximum amplitude of secondary current in a steady state.*

167. Transformation of Partial Resonance Relations S and P into Forms Involving Wavelengths.—As most radiotelegraphic frequency measurements are made in terms of wavelengths, it is proposed to make certain obvious transformations to express equations (8) and (9) in terms of ratios of wavelengths.

It will be remembered that the wavelength λ corresponding to a period T, of angular velocity ω, has been defined by the equation

$$\lambda = cT = 2\pi c/\omega \tag{10}$$

where c is the velocity of light in free space (in meters per second, if λ is in meters and T in seconds).

We have also used in previous chapters the idea of an Undamped Wavelength of a circuit, which ordinarily differs but slightly from the free wavelength λ of the circuit, in that the Undamped Wavelength, designated by a Greek Capital Lambda Λ, is defined as

$$\Lambda = 2\pi c/\Omega \tag{11}$$

The undamped wavelength Λ of a circuit is the wavelength that the circuit would have if its resistance were removed without changing the inductance and capacity of the circuit.

Giving to equation (11) subscripts 1 and 2, and dividing it into (10) we have

$$\Omega_1/\omega = \lambda/\Lambda_1, \qquad \Omega_2/\omega = \lambda/\Lambda_2 \tag{12}$$

In terms of the ratios of wavelengths, equations (8) and (9) may be written

$$\left(1 - \frac{\lambda^2}{\Lambda_1^2}\right)\left(1 - \frac{\lambda^2}{\Lambda_2^2}\right) = \tau^2 - \eta_1^2\left\{\frac{1 - \lambda^2/\Lambda_2^2}{1 - \lambda^2/\Lambda_1^2}\right\} \tag{13}$$

(Partial Resonance Relation S)

and

$$\left(1 - \frac{\lambda^2}{\Lambda_2{}^2}\right)\left(1 - \frac{\lambda^2}{\Lambda_1{}^2}\right) = \tau^2 - \eta_2{}^2 \left\{\frac{1 - \lambda^2/\Lambda_1{}^2}{1 - \lambda^2/\Lambda_2{}^2}\right\} \tag{14}$$

(Partial Resonance Relation P)

In these equations λ is the wavelength of the impressed e.m.f., Λ_1 and Λ_2 are the undamped wavelengths of Circuits I and II respectively. In applying (14) Λ_1 alone is supposed to be varied in obtaining the maximum of amplitude of secondary current. In applying (13) Λ_2 alone is supposed to be varied in obtaining the maximum amplitude of secondary current.

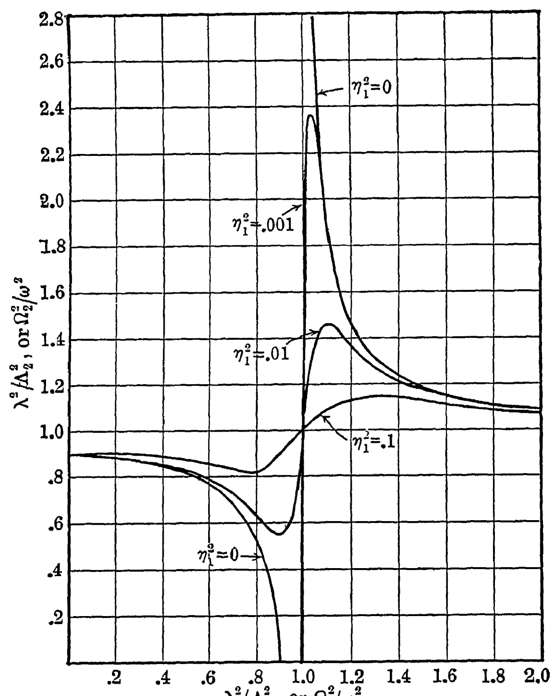

FIG. 3.—Resonant values of λ^2/Λ_2^2 for various values of λ^2/Λ_1^2.

168. Examination of the Partial Resonance Relation S in a Numerical Case.—We shall now take a numerical case in which τ and $\eta_1{}^2$ are given, and shall employ the Partial Resonance Relation S, in the form of equation (13), to determine the value of λ^2/Λ_2 that is required, for various values of λ^2/Λ_1, in order to produce a maximum of amplitude of secondary current.

We shall take, in the example, $\tau = 0.30$, and shall give to η_1^2 the four values 0, 0.001, 0.01, and 0.1. Computed numerical values are contained in Table I.

Where the numbers are omitted near the middle of the table, the values of λ^2/Λ_2^2 are given as negative by the formula, and are therefore impossible of realization, because they would make Λ_2 imaginary.

Table I.—Resonant Values of $(\lambda/\Lambda_2)^2$ for Various Values of $(\lambda/\Lambda_1)^2$. Given $\tau = 0.30$, and Given Four Different Values of η_1^2 Following Partial Resonance Relation S

$(\lambda/\Lambda_1)^2$	$(\lambda/\Lambda_2)^2$ for			
	$\eta_1^2 = 0$	$\eta_1^2 = 0\ 001$	$\eta_1^2 = 0\ 01$	$\eta_1^2 = 0\ 1$
0 0	0 910	0 910	0 911	0.918
0 2	0.888	0.888	0 889	0.903
0 4	0 850	0.850	0 854	0 883
0 6	0 775	0 776	0 789	0 862
0 8	0 550	0 561	0 640	0 812
0 9	0 100	0.182	0 550	0 919
0 95	.	. ..	0 640	0 955
0 97	. .	. ..	0 752	0 973
0 98	....	.....	0 827	0.982
0 99	..	0 181	0 911	0 991
1 00	. ..	1.000	1 000	1 000
1 02	4 500	2 290	1 170	1 017
1.03	3 000	2 360	1 250	1 027
1 05	2 800	2 290	1 360	1 045
1 1	1 900	1 818	1 450	1 082
1 2	1 450	1.439	1 360	1 129
1.3	1.300	1 297	1 270	1 143
1 4	1 225	1 224	1 212	1 139
1 5	1 180	1 179	1 176	1 129
1 6	1 150	1.150	1 146	1 117
1.8	1.112	1 112	1 111	1 097
2 0	1 090	1 090	1 089	1 082
2.5	1.060	1 060	1 060	1 057
3 5	1.036	1.036	1.036	1.035
5 0	1.023	1.023	1 023	1.022
10 0	1.010	1 010	1 010	1.010
∞	1 000	1.000	1.000	1.000

The numerical results of Table I are plotted in the curves of Fig. 3, with $(\lambda/\Lambda_1)^2$ as abscissæ and $(\lambda/\Lambda_2)^2$ as ordinates.

For $\eta_1^2 = 0$, the curve is an equilateral hyperbola with axes

at 1,1, as may be deduced directly by making $\eta_1^2 = 0$ in (13). When $\eta_1^2 = 0.001$, the curve practically coincides with the curve for $\eta_1^2 = 0$ except in the interval of abscissæ between 0.9 and 1.1, where it sweeps from the third quadrant up through the point 1,1 and joins with the part of the curve in the first quadrant. At the bottom of the figure between the abscissæ 0.9 and 0.98 this curve for $\eta_1^2 = 0.001$ has a gap in it. In this gap the com-

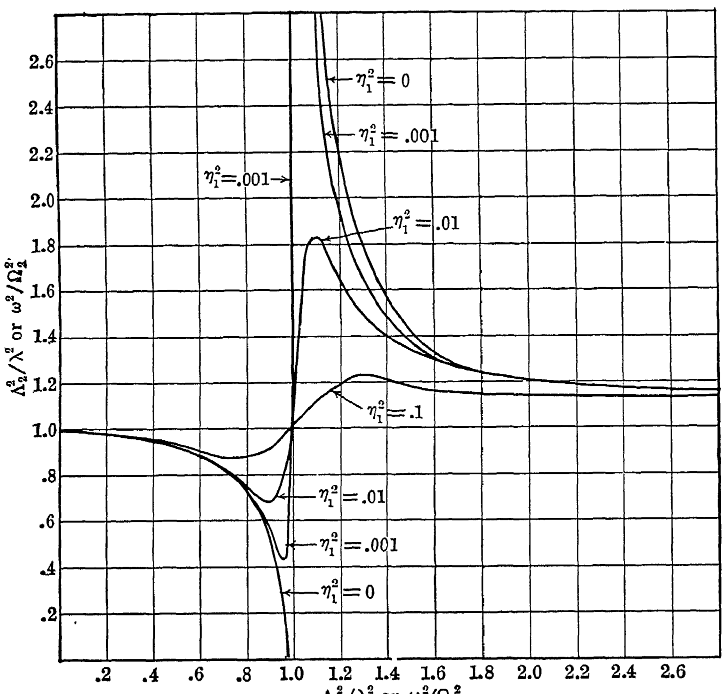

FIG 4.—Resonant values of Λ_2^2/λ^2 for various values of Λ_1^2/λ.

puted values of the ordinates are negative, and the value of Λ_2 is hence imaginary in this region.

The curves for $\eta_1^2 = 0.01$ and 0.1 fall into coincidence with the equilateral hyperbola for large and for small values of abscissæ; but in the neighborhood of the abscissa at 1 they cross over from the first to the third quadrant. The greater the value of η_1^2 the greater the departure of the curve from the equilateral hyperbola.

The whole course of these curves resembles the corresponding

curves in optics, obtained when the index of refraction is plotted against frequency, in the neighborhood of an absorption band.

169. Plot of the Partial Resonance Relation S in the Numerical Case in the Reciprocal Form.—It is deemed worth while to plot the values of the reciprocals of Table I. This will give the curves in the form of $(\Lambda_2/\lambda)^2$ versus $(\Lambda_1/\lambda)^2$. These reciprocals are recorded in Table II, and are plotted in the curves of Fig. 4.

Table II was obtained by taking the reciprocals of all of the numbers within the columns of Table I.

Table II.—Reciprocals of Numbers in Table I

$(\Lambda_1/\lambda)^2$	$(\Lambda_2/\lambda)^2$ for			
	$\eta_1^2 = 0$	$\eta_1^2 = 0\ 001$	$\eta_1^2 = 0\ 01$	$\eta_1^2 = 0\ 1$
∞	1 099	1 099	1 098	1.089
5 0	1.126	1 126	1.125	1.107
2.5	1 176	1 176	1.171	1.133
1 67	1 290	1 289	1.267	1.159
1 25	1 818	1 782	1.563	1.232
1 11	10 000	5.495	1.818	1.088
1 05	....	. ...	1.562	1.047
1 03	. .	. ...	1 330	1.028
1 02	. ..	.....	1.209	1.018
1 01	......	5.525	1.098	1.009
1 00	... ..	1.000	1.000	1.000
0 98	0 222	0.437	0 855	0.983
0 97	0.333	0.424	0.800	0.973
0.95	0.357	0.437	0.735	0.957
0 909	0 526	0.550	0 690	0.924
0 833	0 690	0.695	0.735	0.886
0 769	0 769	0.771	0.787	0.875
0 714	0.816	0.817	0.825	0.878
0.666	0.847	0.848	0.850	0.886
0 625	0.870	0.870	0.873	0.895
0.555	0.899	0.899	0 900	0.912
0 500	0 917	0 917	0.918	0.924
0 400	0.943	0.943	0.943	0.946
0.286	0 965	0.965	0 965	0 966
0.200	0 978	0.978	0 978	0 978
0 100	0.990	0 990	0 990	0 990
0.000	1.000	1 000	1.000	1 000

By reference to Fig. 4, it is seen that in terms of the coordinates of Fig. 4, the curves have lost their symmetry, with the exception of the curve with $\eta_1^2 = 0$, and this has shifted its assymptotes. The equation for this case of $\eta_1^2 = 0$ may be obtained directly as follows:

If the damping term of (13) is negligible, the equation becomes

$$(1 - \lambda^2/\Lambda_1^2)(1 - \lambda^2/\Lambda_2^2) = \tau^2 \tag{15}$$

Performing the indicated multiplications, then multiplying both sides of (15) by $\Lambda_1^2\Lambda_2^2/\lambda^4$, adding $1/(1-\tau^2)^2$, and again factoring, we obtain

$$\left(\frac{\Lambda_1^2}{\lambda^2} - \frac{1}{1-\tau^2}\right)\left(\frac{\Lambda_2^2}{\lambda^2} - \frac{1}{1-\tau^2}\right) = \frac{\tau^2}{(1-\tau^2)^2} \tag{15a}$$

This is seen to be an equilateral hyperbola with asymptotes at

$$(\Lambda_1/\lambda)^2 = 1/(1-\tau^2) \text{ and } (\Lambda_2/\lambda)^2 = 1/(1-\tau^2) \tag{16}$$

Equation (15) *or the alternative equation* (15*a*) *is a statement of the Partial Resonance Relation S in the special case in which* η_1^2 *is negligible. Equation* (16) *is the equation to the asymptotes to the hyperbola* (15).

170. Note on the Partial Resonance Relation P.—We have given in Tables I and II numerical calculations of the partial resonance relation S, and have plotted the results in the curves of Figs. 3 and 4. We shall not here present the corresponding results for the partial resonance relation P, since by the symmetry of equations (13) and (14) it will be evident that the tables and curves will remain as they are except that the subscript 1 will be replaced by 2 and the subscript 2 will be replaced by 1, in order to change the results into values required by the resonance relation P.

171. Effect of Coefficient of Coupling τ on Partial Resonance Relation S in the Case of $\eta_1^2 = 0$.—If the resistance of the primary circuit be so small that η_1^2 is essentially zero, the partial resonance relation S takes the form of equation (15), which is the equation of an equilateral hyperbola in terms of $(\lambda/\Lambda_2)^2$ versus $(\lambda/\Lambda_1)^2$, with asymptotes at

$$(\lambda/\Lambda_2)^2 = 1 = (\lambda/\Lambda_1)^2 \tag{17}$$

A series of such curves computed for different values of τ^2 are plotted in Fig. 5. The computed values are contained in Table III.

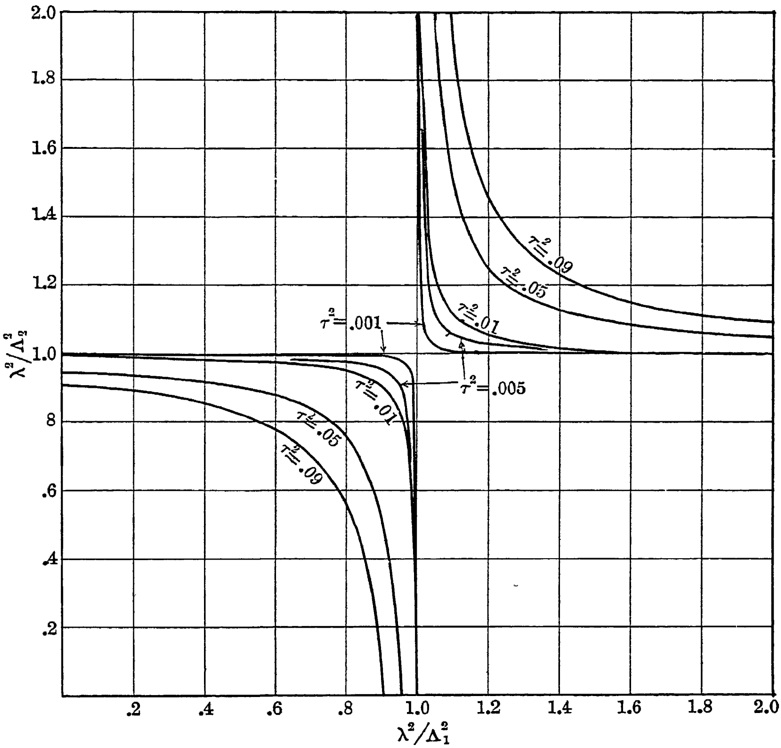

FIG. 5—Showing relation of $\Lambda_2{}^2$ to $\Lambda_1{}^2$ for resonance relation S with various values of τ^2, and with $\eta_1 = 0$.

As may be seen from the equation (15) and from the numerical results, as τ^2 is made smaller and smaller, the equilateral hyperbola approaches the asymptotes, and in case $\tau^2 = 0$, the hyperbola becomes two straight lines coincident with the asymptotes.

Table III.—Resonant Values of $(\lambda/\Lambda_2)^2$ for Various Values of $(\lambda/\Lambda_1)^2$ and Various Values of τ^2, According to Partial Resonance Relation S. Given $\eta_1^2 = 0$

$(\lambda/\Lambda_1)^2$	$(\lambda/\Lambda_2)^2$ for				
	τ^2 = 0 001	τ^2 = 0 005	τ^2 = 0 01	τ^2 = 0 05	τ^2 = 0 09
0 0	0 999	0 995	0 99	0 95	0 91
0 2	0 999	0 994	0 99	0 93	0 88
0 4	0 998	0 992	0 98	0 92	0 85
0 6	0 998	0 985	0 97	0 88	0 78
0 8	0 995	0 978	0 95	0 75	0 55
0 9	0 990	0 950	0 90	0 50	0 10
0 95	0 980	0 900	0 80	0 00	
0 97	0.967	0 835	0.67		
0.98	0 950	0 750	0 50		
0 99	0 900				
1 00					
1 02	1 050	1.250	1 50	3 50	4 50
1 03	1 033	1 165	1 33	2 66	3 00
1 05	1 020	1 100	1 20	2 00	2 80
1 1	1 010	1 050	1 10	1 50	1 90
1 2	1 005	1 025	1 05	1 25	1 45
1 3	1 003	1 016	1 03	1 17	1 30
1 4	1 002	1 012	1 02	1 13	1 23
1 5	1 002	1 010	1 02	1 10	1 18
1 6	1 002	1 008	1 016	1 08	1 15
1 8	1 001	1 006	1 012	1 06	1 11
2 0	1 001	1 005	1 010	1 05	1.09
2 5	1 001	1 003	1 006	1 03	1 06
3 5	1 000	1 002	1 004	1 02	1.04
5 0	1 000	1 001	1 002	1.01	1 02
10 0	1 000	1 000	1 001	1.005	1 01
∞	1.000	1.000	1 000	1 000	1.00

II. OPTIMUM RESONANCE RELATION AS SUFFICIENT COUPLING

172. Case of Sufficient Coupling. Equations for Optimum Resonance in Terms of Angular Velocities.—Let us next examine what we have called in the preceding chapter *the optimum resonance relation,* which is the condition for a maximum maximum of secondary current in the steady state under the action of an impressed sinusoidal e.m.f. The coupling is called *sufficient coupling* whenever the mutual inductance between the two circuits is large enough to make

$$M^2\omega^2 > R_1R_2.$$

The equations for the optimum resonance relation under this condition has been given in suitable form in equations (52) of the preceding chapter. If in these equations we replace X_1 and X_2 by their customary values, and if further we introduce the subscript "*opt*" to designate the optimum relation, we have

$$\left.\begin{aligned} \left\{L_1\omega - \frac{1}{C_1\omega}\right\}_{\text{opt}} &= \pm\sqrt{\frac{R_1}{R_2}(M^2\omega^2 - R_1R_2)} \\ \text{and} \qquad \left\{L_2\omega - \frac{1}{C_2\omega}\right\}_{\text{opt}} &= \pm\sqrt{\frac{R_2}{R_1}(M^2\omega^2 - R_1R_2)} \end{aligned}\right\} \tag{18}$$

where we must use the same sign in both equations to obtain a consistent simultaneous pair of values. This follows from the fact given in equation (51), Chapter XI, that the ratio of X_2 to X_1 must be positive.

If now we divide both sides of (18) by $L_1\omega$, or $L_2\omega$, as required, and use the abbreviations given in (5), (6) and (7), we obtain

$$1 - \left(\frac{\Omega_1^2}{\omega^2}\right)_{\text{opt}} = \pm\,\eta_1\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1} \tag{19}$$

$$1 - \left(\frac{\Omega_2^2}{\omega^2}\right)_{\text{opt}} = \pm\,\eta_2\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1} \tag{20}$$

These equations give the optimum values of the undamped angular velocities Ω_1 and Ω_2 relative to the incident angular velocity ω. These optimum values are values that produce a maximum maximum secondary current amplitude. The equations apply to the case of sufficient coupling, for which

$$M^2\omega^2 > R_1R_2, \; i.e., \; \tau^2 > \eta_1\eta_2 \tag{21}$$

173. The Optimum Resonance Relation in Terms of Wave-lengths, at Sufficient Coupling.—If, in equations (19) and (20) we replace the ratios of angular velocities by the reciprocals of the corresponding ratios of wavelengths, in accordance with equations (12), and make certain evident transformations, we obtain

$$\left(\frac{\Lambda_1}{\lambda}\right)^2_{\text{opt.}} = \frac{1}{1 \pm \eta_1\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1}} \tag{21}$$

$$\left(\frac{\Lambda_2}{\lambda}\right)^2_{\text{opt.}} = \frac{1}{1 \pm \eta_2\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1}} \tag{22}$$

where, for a consistent pair of values, both equations must have the same sign before the radicals.

These resonance relations (21) *and* (22) *are optimum provided*

$$\tau^2 > \eta_1\eta_2 \tag{23}$$

174. Calculation of the Optimum Resonance Relation in Certain Numerical Cases.—In order to facilitate the optimum values of Λ_1/λ and Λ_2/λ, let us extract the square root of (21) and (22) and write the results in the form

$$\left(\frac{\Lambda_1}{\lambda}\right)_{\text{opt}} = \frac{1}{\sqrt{1 \pm \varphi_1}}, \text{ where } \varphi_1 = \eta_1\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1} \tag{24}$$

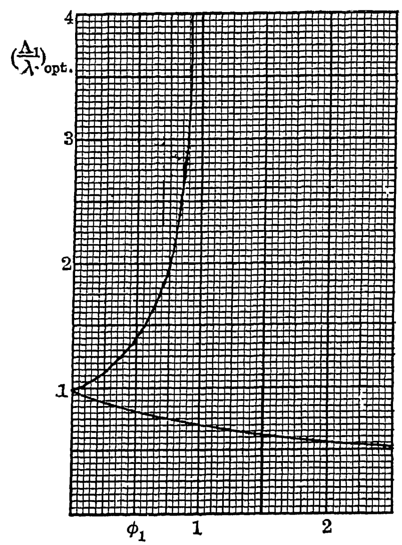

FIG. 6.—Auxiliary curve to assist in calculation of optimum resonance adjustment. φ_1 is defined in (24). These curves give also optimum values of Λ_2/λ if φ_1 is replaced by φ_2 defined in (25).

$$\left(\frac{\Lambda_2}{\lambda}\right)_{\text{opt.}} = \frac{1}{\sqrt{1 \pm \varphi_2}}, \text{ where } \varphi_2 = \eta_2\sqrt{\frac{\tau^2}{\eta_1\eta_2} - 1} \tag{25}$$

provided

$$\tau^2 > \eta_1\eta_2.$$

Table IV gives computed values of $(\Lambda_1/\lambda)_{\text{opt}}$ for various values assumed for φ_1.

The values from this table are plotted in the curves of Fig. 6, with $(\Lambda_1/\lambda)_{opt}$ as ordinates and φ_1 as abscissæ. The lower curve was obtained by using the + sign within the radical of (24), and the upper curve was obtained by using the minus sign in that radical. Note that the same figure may be employed to obtain the values of $(\Lambda_2/\lambda)_{opt}$ for given values of φ_2. To obtain a consistent pair of optimum values, if the upper or lower curve is used to determine Λ_1 the same curve must be used to obtain Λ_2.

Table IV.—Values of $(\Lambda_1/\lambda)_{opt}$. Corresponding to Different Values of φ_1. Computed from Equation (24)

φ_1	$(\Lambda_1/\lambda)_{opt.}$	
	Using + sign	Using − sign
0 0	1 000	1.000
0 1	0.953	1.054
0 2	0.953	1 118
0 3	0 877	1.196
0 4	0.845	1.292
0 5	0 817	1.414
0 6	0.791	1.581
0 7	0 767	1.825
0 8	0.746	2.236
0 9	0.725	3.162
1 0	0.707	Infinite
1 1	0.690	Imaginary
1 2	0.675	Imaginary
1 3	0 660	Imaginary
1 4	0.646	Imaginary
1.5	0 632	Imaginary
1.6	0.620	Imaginary
1 7	0.608	Imaginary
1 8	0 597	Imaginary
1 9	0.587	Imaginary
2 0	0.577	Imaginary
2 1	0 568	Imaginary
2 2	0 559	Imaginary
2 3	0 550	Imaginary
2 4	0.542	Imaginary
2.5	0.535	Imaginary
2 6	0.527	Imaginary

As an example of the manner of using the auxiliary curves of

Fig. 6, in calculation of optimum values of Λ_1 and Λ_2, let us take a special case.

Suppose $\tau^2 = 0.30$ and $\eta_1 = 0.1$, let us give various values to η_2 and compute the corresponding optimum wavelength adjustments, with the results recorded in Table V.

In compiling this table the values of φ_1 and φ_2 corresponding to various values of η_2 were calculated by equations (24) and (25). The corresponding wavelength ratios were then taken from the curve of Fig. 6.

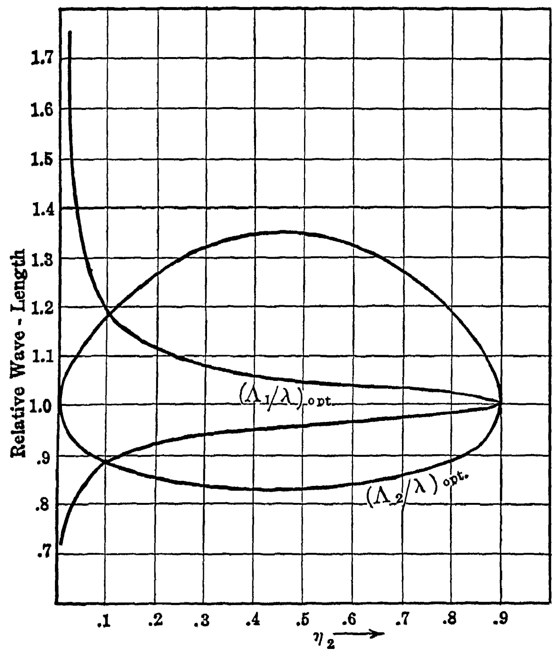

FIG. 7.—Relation of optimum wavelength adjustment to damping in circuit II, for given values of η_1 and τ $(\eta_1 = 0.1, \tau = 0.30)$.

The results contained in Table V are plotted in Fig. 7. In the same way the optimum resonance relations for various values of τ and of η_1 may also be obtained, but the single example here computed and plotted serves to show the manner in which the damping constants contribute to determine the optimum resonance adjustment of the two circuits, with the given coefficient of coupling.

Table V.—Computation of Optimum Resonance Values in a Special Case, in which

$$\tau^2 = 0.30$$
$$\eta_1 = 0.1$$
$$\eta_2 = \text{Various Values}$$

η_2	φ_1	φ_2	$\left(\frac{\Lambda_1}{\lambda}\right)$ opt	$\left(\frac{\Lambda_2}{\lambda}\right)$ opt	$\left(\frac{\Lambda_1}{\lambda}\right)$ opt.	$\left(\frac{\Lambda_2}{\lambda}\right)$ opt.
0 01	0 943	0 094	0 720	0 955	4 07	1 053
0 02	0 662	0 133	0 777	0.940	1 73	1 070
0 03	0 539	0 162	0.806	0.930	1 47	1 090
0 04	0 464	0 186	0 827	0.917	1.37	1 110
0 05	0 412	0.206	0.840	0 910	1 30	1 123
0 06	0.374	0.224	0 850	0.905	1 26	1 135
0 07	0 345	0.242	0 862	0 900	1 23	1 148
0 08	0 319	0.255	0 870	0 895	1 205	1 152
0 09	0.300	0 270	0 875	0.890	1.196	1 168
0 1	0 282	0 282	0.882	0.882	1 180	1 180
0 2	0.187	0.374	0.917	0 853	1 110	1.262
0 3	0 141	0 423	0.935	0.837	1 078	1 310
0 4	0 112	0 448	0.950	0 832	1 060	1 345
0 5	0.089	0 445	0.960	0 830	1 048	1 340
0.6	0.071	0 426	0.970	0.837	1 036	1 320
0 7	0.054	0.378	0.975	0.852	1.032	1 262
0 8	0 035	0.280	0.980	0.885	1 010	1 175
0 9	0 000	0 000	1.000	1.000	1 000	1.000

Either pair of values under a brace is to be employed simultaneously for optimum resonance.

175. General Facts Regarding the Optimum Resonance Relation with Coupling Sufficient.—From the special example just treated and from the equations (21) and (22) the following important facts are apparent in the case of sufficient coupling as defined by the inequality

$$\tau^2 > \eta_1\eta_2.$$

1. With given values of the coefficient of coupling and the damping constants of the two circuits the adjustment for a max. max. value of secondary current is in general doubly valued. One may in general get best resonance either by setting both wavelengths appropriately longer than the incident waves, or by setting both circuits to a wavelength appropriately shorter than the incident waves.

2. The adjustment for optimum resonance is materially in-

fluenced by the resistances of the two circuits, so that, in general, with fixed incident waves, if one tunes a radiotelegraphic system of the coupled type to resonance, with the use of a given detector, and then changes to a detector of different resistance, it is necessary to shift the wavelength of both the circuits in order to bring the system back to optimum adjustment.

3 With fixed values of the damping factors, and provided $\tau^2 > \eta_1\eta_2$, the proper adjustment for a maximum secondary current is materially influenced by the coefficient of coupling τ, and every change of τ requires a readjustment of the wavelengths of both of the circuits of the coupled system.

III. CURRENT AMPLITUDE AT OPTIMUM RESONANCE

176. General Value of Secondary Current Amplitude.—In equation (33) of the preceding chapter we have the general expression for the secondary current amplitude, and this expression, in view of (35) of the same chapter, may be written

$$I_2 = \frac{M\omega E}{\sqrt{Z_1^2Z_2^2 + M^4\omega^4 + 2M^2\omega^2(R_1R_2 - X_1X_2)}} \tag{26}$$

where X_1, X_2, Z_1, and Z_2 are the ordinary abbreviations for the reactances and impedances, defined as follows·

$$X_1 = L_1\omega - 1/C_1\omega, \qquad X_2 = L_2\omega - 1/C_2\omega \tag{27}$$

$$Z_1^2 = R_1^2 + X_1^2, \qquad Z_2^2 = R_2^2 + X_2^2 \tag{28}$$

In these equations ω is the angular velocity of the impressed e.m.f ; E is the amplitude of impressed e m.f.; and M is the mutual inductance between the two circuits. I_2 is the amplitude of the secondary current for any values whatever of the constants of the circuits.

177. Current Amplitude in Secondary Circuit at Optimum Resonance, with Coupling Sufficient.—We have also seen in the preceding chapter that if

$$M^2\omega^2 > R_1R_2 \tag{29}$$

or, otherwise expressed, if

$$\tau^2 > \eta_1\eta_2 \tag{30}$$

the secondary current amplitude obtained at optimum resonance is, by Chapter XI equation (53),

$$I_{2_{\text{max. max.}}} = \frac{E}{2\sqrt{R_1R_2}} \tag{31}$$

In this case the amplitude of secondary current is independent of the coefficient of coupling provided only (29), *or* (30), *is fulfilled.*

178. Current Amplitude in Secondary Circuit at Optimum Resonance with Coupling Deficient.—The coupling is called deficient whenever

$$\tau^2 < \eta_1\eta_2 \tag{32}$$

Under this condition, by equation (49) Chapter XI, the value of the amplitude of secondary current is

$$I_{2_{\text{max. max}}} = \frac{M\omega E}{R_1R_2 + M^2\omega^2} \tag{33}$$

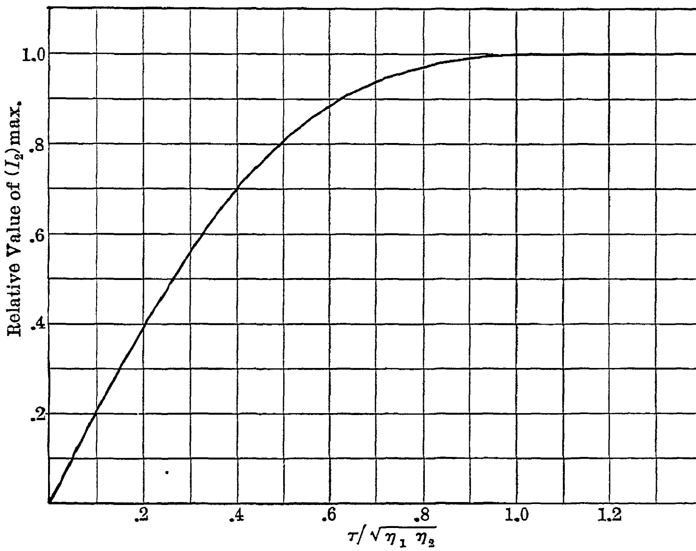

FIG. 8.—Relative values of max. max. secondary current for different values of $\tau/\sqrt{\eta_1\eta_2}$.

In terms of the ratio constants τ, η_1 and η_2, defined in (5), (6), and (7) this can be written

$$I_{2_{\text{max max}}} = \frac{E}{\sqrt{R_1R_2}\left\{\dfrac{\sqrt{\eta_1\eta_2}}{\tau} + \dfrac{\tau}{\sqrt{\eta_1\eta_2}}\right\}} \tag{34}$$

In this case the amplitude of current depends upon the ratio of τ *to* $\sqrt{\eta_1\eta_2}$.

Table VI following contains a series of values of relative amplitude of $I_{2\text{max max}}$ for various values of the ratio $\tau/\sqrt{\eta_1\eta_2}$. These results are plotted in the curve of Fig. 8.

In this table and curve the relative amplitude of secondary current is arbitrarily designated as unity for $\tau^2 = \eta_1\eta_2$.

Table VI.—Relative Values of $I_{2\max\ \max}$ for Different Values of the Ratio $\tau/\sqrt{\eta_1\eta_2}$

$\tau/\sqrt{\eta_1\eta_2}$	Relative values of $I_{2\max}$ max.
>1	1 000
1 00	1 000
0 90	0 995
0 80	0 974
0 70	0.937
0.60	0 880
0 50	0 800
0 40	0 690
0 30	0 551
0 20	0 385
0 10	0.198

IV. ON THE SHARPNESS OF RESONANCE AND THE POSSIBILITY OF AVOIDING INTERFERENCE

179. Ratio of Interference.—If we have an electromagnetically coupled receiving station of the form of Fig. 1, and if we set our receiving station in the optimum resonance condition for a given desired wave of angular velocity ω_0, we shall receive from this wave an amplitude of current $I_{2\max\ \max}$ given by equation (31), if the coupling is *sufficient*, and by (33), if the coupling is *deficient;* where E is the amplitude of e.m.f. impressed by the wave of ω_0, and where the ω of (33) is to be replaced by ω_0.

If now at the same time someone else is sending electric waves with a different angular velocity ω, and is at such a distance from us as to impress an equal amplitude of e.m.f., we shall receive from him an amount of interfering current given by (26).

Let us now take the ratio of the interfering current to the desired current, and call this ratio **the ratio of interference,** *indicated by* Y.

Then, on forming the indicated ratio, we have:

If the Coupling is Sufficient (*i.e., if* $M^2\omega_0^2 > R_1R_2$)

$$Y = \frac{[I_2]\omega}{[I_2]\omega_0} = \frac{1}{\sqrt{\dfrac{Z_1^2Z_2^2 + M^4\omega^4 + 2M^2\omega^2(R_1R_2 - X_1X_2)}{4M^2\omega^2R_1R_2}}} \tag{35}$$

If now we designate by η_{10} and η_{20} the values that η_1 and η_2 have when $\omega = \omega_0$, we have

$$\eta_{10} = R_1/L_1\omega_0, \qquad \eta_{20} = R_2/L_2\omega_0 \tag{36}$$

If also we let

$$\varphi_0 = \sqrt{\frac{\tau^2}{\eta_{10}\eta_{20}} - 1} \tag{37}$$

we shall have, by the fact that the circuits are in optimum resonance for angular frequencies ω_0, by (19) and (20), the additional equations

$$1 - \frac{\Omega_1^2}{\omega_0^2} = \pm\, \eta_{10}\varphi_0, \text{ and } 1 - \frac{\Omega_2^2}{\omega_0^2} = \pm\eta_{20}\varphi_0 \tag{38}$$

In terms of these ratio constants, we may change the form of X_1, as follows:

$$X_1 = L_1\omega - 1/C_1\omega = L_1\omega\left(1 - \frac{\Omega_1^2}{\omega^2}\right)$$

$$= L_1\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\frac{\Omega_1^2}{\omega_0^2}\right)$$

which by (38) gives

$$X_1 = L_1\omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \pm \eta_{10}\varphi_0\frac{\omega_0}{\omega}\right)$$

$$= L_1\omega_0 u_1 \text{(say)} \tag{39}$$

Likewise

$$X_2 = L_2\omega_0 u_2 \text{(say)} \tag{40}$$

where

$$u_1 = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \pm \frac{\eta_{10}\varphi_0\omega_0}{\omega},\ u_2 = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \pm \frac{\eta_{20}\varphi_0\omega_0}{\omega} \tag{41}$$

If now we divide the numerator and denominator of the fraction under the radical in (35) by $L_1^2L_2^2\omega_0^4$, and make use of the abbreviations above given, we obtain

$$Y = \frac{1}{\sqrt{\dfrac{(\eta_{10}^2 + u_1^2)(\eta_{20}^2 + u_2^2) + \tau^4\dfrac{\omega^4}{\omega_0^4} + 2\tau^2\dfrac{\omega^2}{\omega_0^2}(\eta_{10}\eta_{20} - u_1u_2)}{4\tau^2\eta_{10}\eta_{20}\omega^2/\omega_0^2}}} \tag{42}$$

Equation (42) can be otherwise factored so as to give

$$Y = \frac{1}{\sqrt{\dfrac{(\eta_{10}u_2 + \eta_{20}u_1)^2 + \left(\tau^2\dfrac{\omega^2}{\omega_0^2} + \eta_{10}\eta_{20} - u_1u_2\right)^2}{4\tau^2\eta_{10}\eta_{20}\omega^2/\omega_0^2}}} \tag{43}$$

In equation (43) Y *is the ratio of interference at sufficient coupling. It is the ratio of the secondary current produced by the interfering signal of angular velocity* ω *to the secondary current produced by the desired signal of angular velocity* ω_0. *The two signals are supposed to be of such intensity as to impress equal amplitudes of e.m.f. on the receiving antenna.*

We shall next write out a similar equation for the ratio of interference at deficient coupling.

If the Coupling is Deficient (*i.e., if* $M^2\omega^2 < R_1R_2$),
we obtain Y by dividing (26) by (33), with ω in (33) replaced by ω_0. This gives

$$Y = \frac{1}{\sqrt{\dfrac{Z_1^2Z_2^2 + M^4\omega^4 + 2M^2\omega^2(R_1R_2 - X_1X_2)}{(R_1R_2 + M^2\omega_0^2)^2\omega^2/\omega_0^2}}} \tag{44}$$

Now we introduce the condition that the constants of the circuits are such that the system is in optimum resonance (with deficient coupling) with the angular velocity ω_0; that is, by (48) Chapter XI,

$$X_1 = 0 = X_2, \qquad \text{at } \omega = \omega^0.$$

These last two equations give

$$1 - \Omega_1^2/\omega_0^2 = 0 = 1 - \Omega_2^2/\omega_0^2 \tag{45}$$

Dividing numerator and denominator of (44) by $L_1^2L_2^2\omega_0^4$, subject to the condition (45), we obtain from (44)

$$Y = \frac{1}{\sqrt{\dfrac{(\eta_{10}^2 + v^2)(\eta_{20}^2 + v^2) + \tau^4\dfrac{\omega^4}{\omega_0^4} + 2\tau^2\dfrac{\omega^2}{\omega_0^2}(\eta_{10}\eta_{20} - v^2)}{\dfrac{\omega^2}{\omega_0^2}(\eta_{10}\eta_{20} + \tau^2)^2}}} \tag{46}$$

where

$$v = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \tag{46}$$

Equation (46) may be otherwise factored so as to take the form

Equation (47) gives the ratio of interference Y at deficient coupling. The quantity Y is the ratio of the secondary current produced by the interfering signal of angular velocity ω to the secondary current produced by the desired signal of angular velocity ω_0, when the signals are such as to impress equal amplitudes of e.m.f. on the receiving antenna.

180. Tables and Curves Showing the Ratio of Interference in a Typical Case.—Values calculated for the ratio of interference Y in a specific case are contained in Tables VII and VIII. These two tables of values were obtained with

$$\eta_{10} = 0.03, \ \eta_{20} = 1.00, \ \tau^2 = \text{various values} \qquad (48)$$

The various values employed for τ^2 are indicated in the headings to the columns in the Tables.

Graphs of the values given in these tables are exhibited in Figs. 9 to 12. The tables and curves employ as parameter the value of λ/λ_0 $(=\omega_0/\omega)$ where λ_0 is the wavelength of the desired signal, and λ the wavelength of the interfering signal.

In all three of the figures the black dots are values obtained with the case of critical coupling ($\tau^2 = \eta_{10}\eta_{20} = 0.03$).

Table VII.—Values of the Ratio of Interference Y at Sufficient Coupling for Different Values of Relative Incident Wavelengths, and Different Coefficients of Coupling τ. Given η_{10} = 0.03, η_{20} = 1.00

λ/λ_0	Y for							
	$\tau^2 = 0.51$		$\tau^2 = 0.30$		$\tau^2 = 0.15$		$\tau^2 = 0.06$	
	(+)	(−)	(+)	(−)	(+)	(−)	(+)	(−)
0.87	0.319	0.171	0.293	0.240	0.256	0.247	0.232	0.278
0.909	0.411	0.296	0.387	0.312	0.353	0.334	0.327	0.356
0.952	0.638	0.475	0.619	0.497	0.577	0.524	0.548	0.553
0.98	0.886	0.779	0.876	0.795	0.871	0.814	0.849	0.833
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.02	0.875	0.833	0.865	0.767	0.847	0.776	0.841	0.805
1.05	0.580	0.400	0.557	0.416	0.567	0.433	0.530	0.468
1.10	0.320	0.198	0.309	0.225	0.297	0.215	0.298	0.230
1.15	0.209	0.125	0.203	0.129	0.198	0.134	0.204	0.144

The columns headed (+) were obtained by using the plus sign in the expressions for u_1 and u_2 (41), and belong to the long-wave optimum adjustment of the receiving circuits; while the columns headed (−) were obtained by using the minus sign in equation (41) and belong to the short-wave optimum adjustment of the two receiving circuits.

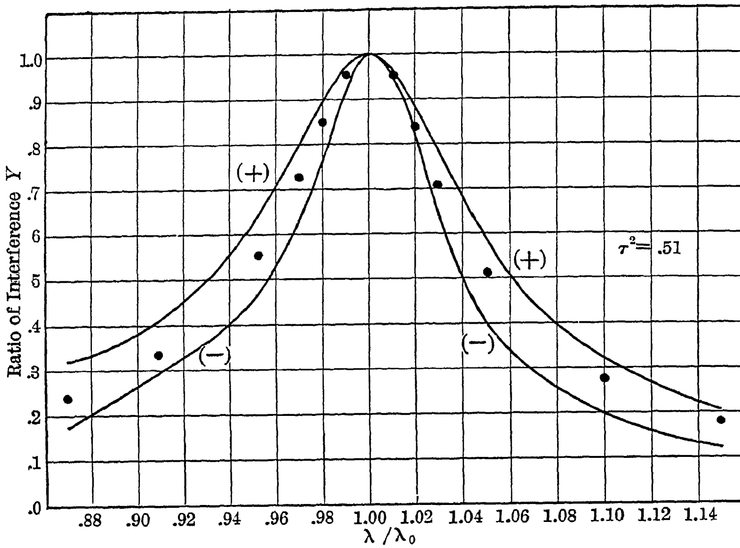

FIG. 9.—Ratio of interference. λ_0 = wavelength of desired signal. λ = interfering wavelength. Black dots = values obtained at critical coupling ($\tau^2 = 0.03$) Sign (+) designates use of long-wave optimum adjustment; sign (−) designates use of short-wave optimum adjustment. Given $\eta_{10} = 0.03$, $\eta_{20} = 1.00$.

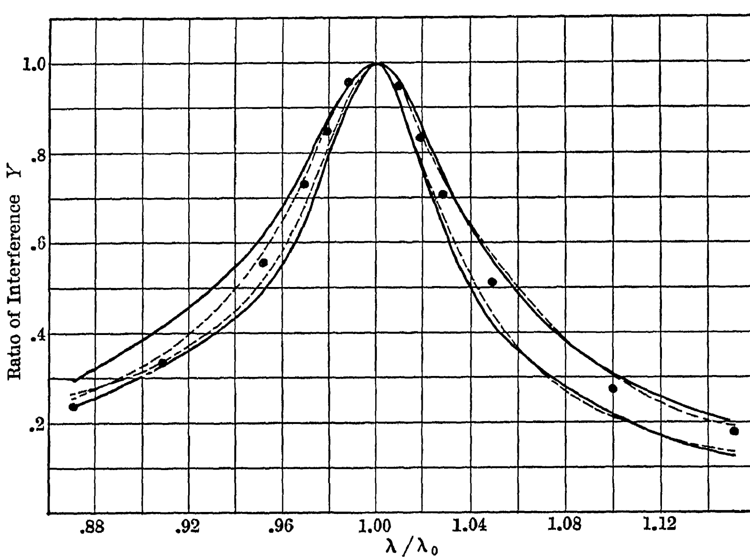

FIG. 10—Ratio of interference. Heavy lines for $\tau^2 = 0.30$. Dotted lines for $\tau^2 = 0.15$. Top curves using long-wave optimum adjustment; bottom curves using short-wave optimum adjustment. Black dots obtained at critical coupling. Given $\eta_{10} = 0.03$; $\eta_{20} = 1.00$.

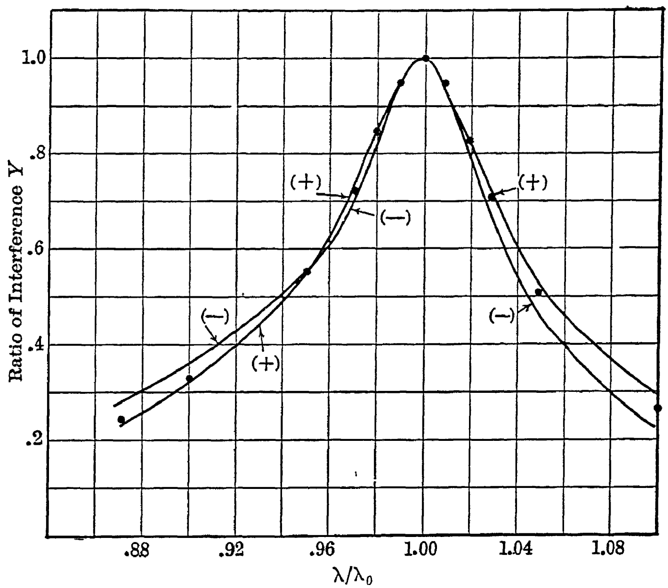

FIG. 11.—Same as Fig. 9, except that $\tau^2 = 0.06$.

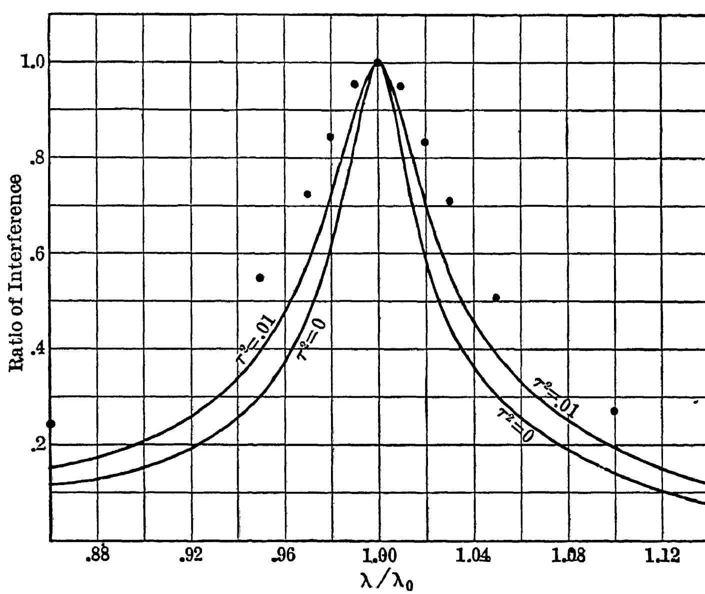

FIG. 12.—Ratio of interference at deficient coupling for $\tau^2 = 0.01$ and $\tau^2 = 0$. Black dots obtained at critical coupling. Given $\eta_{10} = 0.03$, $\eta_{20} = 1.00$.

Table VIII.—Similar to Table VII, but with Deficient Coupling

λ/λ_0	Y for		
	$\tau^2 = 0\ 03$	$\tau^2 = 0\ 01$	$\tau^2 = 0\ 001$
4 00	0 00102	0 00068	0 00052
3 00	0 00263	0 00177	0 00130
2 00	0 0111	0 0074	0 00555
1 50	0 0370	0 0246	0 0183
1 25	0 0984	0 0653	0 0483
1 15	0 180	0 119	0 0888
1 10	0 274	0 184	0 138
1 05	0 510	0 380	0 279
1 03	0 707	0 543	0 435
1 02	0 833	0 700	0 588
1 01	0 950	0 890	0 825
1 00	1 000	1 000	1 000
0 99	0 952	0 899	0 841
0 98	0 848	0 721	0 612
0 97	0 728	0 571	0 461
0 952	0 552	0 397	0 306
0 909	0 333	0 222	0 167
0 870	0 239	0 157	0 117
0 800	0 156	0 103	0 0760
0 667	0 0860	0 0558	0 0413
0 500	0 0460	0 0300	0 0222
0 333	0 0242	0 0159	0 0118
0 250	0 0168	0.0108	0 00820

By reference to Fig. 12, one sees that *with deficient coupling* a decrease of the coefficient of coupling always diminishes the interference for any wavelength of the interfering signal.

With the *coupling sufficient,* as displayed in Figs. 9, 10, and 11, the ratio of interference for a given coefficient of coupling may be either greater or less than the interference with the smaller coefficient of coupling designated as *Critical Coupling.* In this case, with $\tau^2 = 0.03$, the coupling is critical, for then $M^2\omega_0^2 = R_1R_2$, or, otherwise stated, $\tau^2 = \eta_{10}\eta_{20}$.

With the coupling sufficient, the curve for the long-wave tuning in the neighborhood of resonance shows generally a larger interference than the curve of short-wave tuning, but if the range of wavelengths is sufficiently extended the two curves cross and show the reverse condition. Such a crossing point is shown at $\lambda/\lambda_0 = 0.885$ on one of the pairs of curves

of Fig. 10. A mathematical investigation shows that the curve of interference for long-wave tuning always crosses the curve of interference for short-wave tuning at the point given by the equation

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1 - \tau^2}{1 - \eta_{10}\eta_{20} + \tau^2}} \tag{48}$$

V. MAX. MAX. SECONDARY CURRENT AND DETECTOR RESISTANCE

181. At Optimum Resonance with Coupling Sufficient the Total Heat Developed in the Secondary Circuit is Independent of its Resistance.—At Sufficient Coupling; that is, when

$$M^2\omega^2 > R_1R_2, \tag{49}$$

the current obtained at optimum resonance has been found to be

$$I_{2\,\max\,\max} = \frac{E}{2\sqrt{R_1R_2}} \tag{50}$$

which shows the striking property of being independent of the mutual inductance between the circuits, provided only that $M\omega$ is great enough to fulfill the condition for sufficient coupling.

If the resistances of the two circuits are independent of the frequency, the higher the frequency the smaller M can be and yet have (50) fulfilled. For this reason, high-frequency transformers may be coupled much more loosely than corresponding transformers for low frequency, and iron which is used to increase M in low-frequency transformers is not advantageous in high-frequency transformers.

Another very interesting and important fact is the fact that can be obtained from (50) that the heat developed in the secondary circuit at optimum resonance with coupling sufficient is independent of the resistance R_2 of the secondary circuit; for if we multiply the square of the secondary current by R_2, we obtain for the power dissipated in the secondary circuit a quantity independent of R_2.

This means that at optimum resonance with sufficient coupling there is as much heat developed in the secondary circuit when a low-resistance detector is used as when a high-resistance detector is used. If, therefore, the detector is an instrument whose

indications are proportional to the heat developed, a low-resistance detector would be as sensitive as a high-resistance detector if it were not for the fact that a low-resistance detector is a smaller proportion of the total resistance of the secondary circuit.

Similar considerations apply to a detector of the electrodynamometer type. If the deflections of the electrodynamometer are proportional to $n^2I_2^2$, where n is the number of turns of wire in the coil, and if the size of the channel of windings is fixed so that the resistance R of the detector is $\rho l/s$, l and s being the length and cross section of the wire in the coil and ρ the specific resistance of the material of the wire, then we have

$$l = 2\pi rn,$$

in which r is the mean radius of the windings; and approximately

$$S = A/n,$$

where A is the area of the channel.

Therefore,

$$R = \frac{2\pi\rho rn^2}{A},$$

or

$$R \propto n^2,$$

whence, if the deflection D is such that

$$D \propto n^2I_2^2,$$

we have

$$D \propto RI_2^2$$

This gives for the circuit containing the electrodynamometer detector the same relations as with the thermal detector above specified.

From the results here obtained, we may draw the following conclusions:

If the detector is to be used in series with the secondary circuit, and if the indications of the detector are proportional to the square of the secondary current times the resistance of the detector, and if the resistance of the remainder of the secondary circuit is inconsiderable in comparison with the resistance of the detector, and if the e.m.f. impressed on the antenna by the incoming waves has an amplitude uninfluenced by the tuning of the secondary circuit, and

if the efficiency of the detector is independent of its resistance, then the indications of the low-resistance detector will be as great as the indications of a high-resistance detector. The low-resistance detector will then be preferred to the high-resistance detector, because resonance with the low resistance is sharper.

This analysis is given in the effort to determine the theoretical limitation upon the choice of a detector for use in series in the secondary circuit of a radiotelegraphic receiving station.

In practice, up to the present time, only detectors of comparatively high resistance are found to be applicable to the reception of weak signals. The reason, in the form of an alternative, is apparent from the analysis here given, to wit:

Either, the detectors of low resistance have a smaller efficiency in the conversion of the oscillatory energy into perceptible indications;

Or, the low-resistance detector by permitting and requiring a larger value of I_2^2 *causes such large reactions on the received antenna current as to modify materially the electromagnetic field of the incident waves.*

The first of these alternative possibilities is a matter for experimentation on the conversion factors of the detectors themselves. The second of the possibilities is a matter for theoretical investigation by the use of Maxwell's Theory of the Electromagnetic Field.

CHAPTER XIII

A GENERAL RECIPROCITY THEOREM IN STEADY-STATE ALTERNATING-CURRENT THEORY WITH APPLICATION TO THE DETERMINATION OF RESONANCE RELATIONS

I. RECIPROCITY THEOREM IN STEADY-STATE ALTERNATING-CURRENT THEORY

182. Statement of the Reciprocity Theorem.—*If we have any system of ironless alternating-current circuits, however complicated, and if we have in the system a sinusoidal impressed e.m.f. applied at any point of the system and an impedanceless ammeter at any other point of the system, the ammeter and e.m.f. are interchangeable without changing the amplitude or phase of the steady-state current through the ammeter.*

This theorem will be proved below.

183. Utility of the Theorem.—With a given system of circuits by making suitable interchanges of ammeter and e.m.f., we may obtain several different expressions for the same current, and may then determine important resonance relations by inspection.

This process has important applications (for example, to telephony and radiotelegraphy) in obtaining steady-state resonance relations in respect to the variables of the system.

184. Example with Two Circuits.—To begin with let us prove the reciprocity theorem for two circuits, called Circuit I and Circuit II, as illustrated in Fig. 1. To make the problem as general as possible, let us suppose the two circuits to be coupled together by having a common conductive part which may contain inductance, L_0, resistance R_0 and capacity C_0, and to be further coupled by having a mutual inductance M in the form of a transformer.

As we go around Circuit I, let

R_1 = the sum of all the resistances in series, including resistances common to both circuits.

L_1 = the sum of all the self-inductances in series including

common inductances and the inductance of that coil of the transformer that is in Circuit I.

$1/C_1$ = the sum of the reciprocals of all the capacities including common capacities.

As we go around the Circuit II, let

R_2, L_2 and $1/C_2$ be the corresponding quantities for Circuit II.

Let R_0, L_0 and $1/C_0$ be the corresponding quantities common to both circuits. These will be called *mutual values.*

Let M be the mutual inductance of the two coils of the transformer with its primary in one circuit and its secondary in the other circuit.

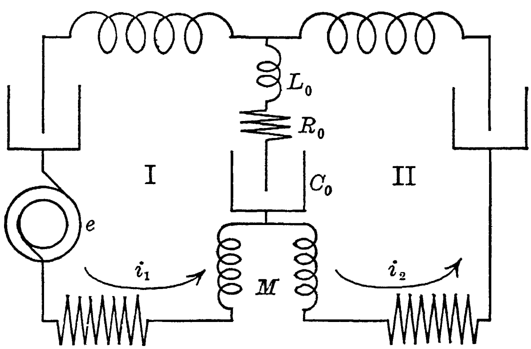

FIG. 1 —Two circuits I and II with involved coupling.

We shall use the real quantities X_1, X_2, Z_1, Z_2 and the complex quantities z_1, z_2 in their ordinary engineering significance. For the common part of the two circuits, we shall let

$$X_0 = L_0\omega - \frac{1}{C_0\omega} \tag{1}$$

$$Z_0 = \sqrt{R_0^2 + X_0^2} \tag{2}$$

and

$$z_0 = R_0 + jX_0 \tag{3}$$

where ω is the angular velocity of the impressed e.m.f.

185. The Differential Equations.—If we take the impressed e.m.f. in the form

$$e = E\cos\omega t \tag{4}$$

we may temporarily replace it by the complex quantity

$$e' = E\epsilon^{j\omega t} \tag{5}$$

and after solving the differential equations take only the real part of the result. If now we let i_1 be the current in those parts of Circuit I that are not common to Circuit II and i_2 the corresponding current in Circuit II, the differential equations, obtained by taking the counterelectromotive force around each circuit and equating it to the impressed electromotive force plus the e.m.f induced from the other circuit, are

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{\int i_1 dt}{C_1} = E\epsilon^{j\omega t} + L_0 \frac{di_2}{dt} + R_0 i_2 + \frac{\int i_2 dt}{C_0} + M \frac{di_2}{dt} \tag{6}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{\int i_2 dt}{C_2} = L_0 \frac{di_1}{dt} + R_0 i_1 + \frac{\int i_1 dt}{C_0} + M\frac{di_1}{dt} \tag{7}$$

186. Steady-state Solution.—To obtain the steady-state solution of these equations (6) and (7), we shall let

$$i_1 = A_1\epsilon^{j\omega t}, \qquad i_2 = A_2\epsilon^{j\omega t} \tag{8}$$

Substituting (8) into (6) and (7), dividing out $\epsilon^{j\omega t}$, and making use of the usual notation, we obtain

$$z_1 A_1 = E + (z_0 + Mj\omega)A_2 \tag{9}$$

$$z_2 A_2 = (z_0 + Mj\omega)A_1 \tag{10}$$

Let us introduce as an abbreviation the complex quantity m defined by the equation

$$m = z_0 + Mj\omega \tag{11}$$

We shall call m the *complex mutual impedance* between the two circuits.

Then equations (9) and (10) may be written

$$z_1 A_1 - mA_2 = E \tag{12}$$

$$z_2 A_2 - mA_1 = 0 \tag{13}$$

187. Proof of Reciprocity Theorem for These Two Circuits.—From equations (11) and (13) we can obtain the value of the current in the circuit II when the e.m.f. is impressed on Circuit I. The vector amplitude of this current obtained by solving the two equations as simultaneous is

$$A_2 = \frac{mE}{z_1 z_2 - m^2} \tag{14}$$

Let us next suppose that the impressed e.m.f. is removed from Circuit I and applied to Circuit II, then the equations corresponding to (12) and (13) are

$$z_1A_1 - mA_2 = 0 \tag{15}$$

$$z_2A_2 - mA_1 = E \tag{16}$$

The solution of these equations as simultaneous gives, for A_1,

$$A_1 = \frac{mE}{z_1z_2 - m^2} \tag{17}$$

If the e.m.f. is applied to Circuit I, the vector amplitude of current in Circuit II is given by (14). *When the same e.m.f. is removed from Circuit I and applied to Circuit II, the vector amplitude of current in Circuit I is given by* (17). *Whence it appears that the ammeter reading both as to amplitude and phase is unchanged by an interchange of ammeter and e.m.f. The definition of m is given in* (11).

188. Proof of Reciprocity Theorem for n Circuits Coupled in Any Way.—Let us suppose that we have n circuits coupled in any way by common conductive portions and by transformers, any or all circuits being coupled with any or all others. Between any two circuits, for example the third and the fifth, let what we have called the *complex mutual impedance m* be

$$m_{35} = z_{35} + M_{35}j\omega \tag{18}$$

where

z_{35} = the vector impedance common to the two circuits,

and

M_{35} = the mutual inductance between them.

Let us now note that the complex mutual impedance is reciprocal, so that

$$m_{35} = m_{53} \tag{19}$$

as may be seen from the manner of its formation (provided there are no distributed capacities in the circuits, such as to make M_{35} different from M_{53}).

If now as before we let the currents in the uncommon portions of the several circuits be

$$i_1 = A_1\epsilon^{j\omega t},\ i_2 = A_2\epsilon^{j\omega t},\ i_3 = A_3\epsilon^{j\omega t}\ \ldots \tag{20}$$

and if we note that every circuit may (or may not) act on every other, the equations formed as a generalization of (12) and (13), and connecting the several coefficients A_1, A_2, A_3 . . . will be

$$\left.\begin{array}{rl}
z_1A_1 - m_{12}A_2 - m_{13}A_3 - m_{14}A_4 \ldots\ldots & = E \\
-m_{21}A_1 + z_2A_2 - m_{23}A_3 - m_{24}A_4 \ldots\ldots\ldots & = 0 \\
-m_{31}A_1 - m_{32}A_2 + z_3A_3 - m_{34}A_4 \ldots\ldots\ldots & = 0 \\
-m_{41}A_1 - m_{42}A_2 - m_{43}A_3 + z_4A_4 \ldots\ldots\ldots & = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & = 0
\end{array}\right\} \quad (21)$$

where the e.m.f. is impressed on Circuit I.

We may now write down the determinant from which can be obtained the vector current amplitude in any one of the circuits. Let us for example form such a determinant for A_3. It is

$$A_3\begin{vmatrix} z_1 & -m_{12} & -m_{13} & -m_{14} & \\ -m_{21} & z_2 & -m_{23} & -m_{24} & \\ -m_{31} & -m_{32} & z_3 & -m_{34} & .. \\ -m_{41} & -m_{42} & -m_{43} & z_4 & . \\ .. & . & ... & .... & \end{vmatrix} = E\begin{vmatrix} -m_{12} & z_2 & -m_{24} & .. \\ -m_{31} & -m_{32} & -m_{34} & . \\ -m_{41} & -m_{42} & z_4 & . \\ ... & .... & ... & .. \end{vmatrix} \quad (22)$$

It will not be necessary to reduce this determinant.

Let us next suppose that the e.m.f. and the ammeter are interchanged. This will put the amplitude E of the applied e.m.f. in the right-hand side of the third of the equations (21) instead of in the first. If we then solve the set of simultaneous equations (21) for A_1 instead of for A_3, we obtain the determinant

$$A_1\begin{vmatrix} \text{same determinant as at left} \\ \text{of (22)} \\ \\ \end{vmatrix} = E\begin{vmatrix} -m_{12} & -m_{13} & -m_{14} & .. \\ z_2 & -m_{23} & -m_{24} & .. \\ -m_{42} & -m_{43} & z_4 & .. \\ .. & . & . & ... \end{vmatrix} \quad (23)$$

It is seen that the determinant on the right-hand side of this equation is the same as the determinant on the right-hand side of (22) except that the rows of the one are the columns of the other. This, however, leaves the two determinants equal. We have then the result that A_1 in (23) is equal to A_3 in (22).

Since the particular circuits employed in this demonstration are any two circuits, we have proved the reciprocity theorem enunciated in the first paragraph of this chapter, for all cases except where the e.m.f. or ammeter is placed in a common member of

the system. The theorem is also true when the e.m.f. or the ammeter is placed in the common member, as the following reasoning with two circuits shows.

188a. Proof of the Reciprocity Theorem When the E.M.F. or the Ammeter is Placed in a Common Member.—For this proof it will be sufficient to take two circuits, as shown in Fig. 2, with the e.m.f. e applied (say) to the common member of the circuits. The e.m.f. will then be impressed on both circuits, but since the two currents are both estimated positive in a clockwise sense, the e.m.f. will aid one of the currents and oppose the other in the common member, so that the equations for the vector current amplitudes become (compare (12) and (13))

$$\left.\begin{aligned} z_1A_1 - mA_2 &= E \\ z_2A_2 - mA_1 &= -E \end{aligned}\right\} \qquad (24)$$

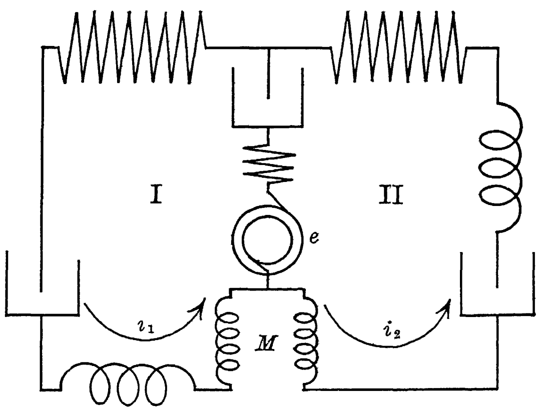

FIG. 2.—Two circuits with e.m f. in commom member.

A solution of these equations for A_1 gives

$$A_1 = \frac{E(z_2 - m)}{z_1z_2 - m^2} \qquad (25)$$

Let us now interchange the e.m.f. and current-measuring apparatus. The amplitude equations then become

$$\left.\begin{aligned} z_1A_1 - mA_2 &= E \\ z_2A_2 - mA_1 &= 0 \end{aligned}\right\} \qquad (26)$$

The current-measuring apparatus is now inserted in the common member so that it measures the instantaneous difference

of the two currents, determined by the vector magnitude $A_1 - A_2$. Let us determine this difference. Equation (26) gives

$$\left.\begin{aligned} A_1 &= \frac{Ez_2}{z_1z_2 - m^2} \\ A_2 &= \frac{Em}{z_1z_2 - m^2} \end{aligned}\right\} \tag{27}$$

so

$$A_1 - A_2 = \frac{E(z_2 - m)}{z_1z_2 - m^2} \tag{28}$$

Equation (25) *gives the vector amplitude of current in Circuit I when the e.m.f. is applied to the common member.*

Equation (28) *gives the vector amplitude of current in the common member when the e.m.f. is applied to Circuit I. The right-hand sides of the two equations are equal. It is seen that the general reciprocity theorem enunciated in the first paragraph is therefore true even when the e.m.f. or ammeter is applied to a common member of the system of circuits.*

II. CURRENT AMPLITUDES IN A CHAIN OF CIRCUITS

Before attempting to use the reciprocity theorem in the determination of resonance relations it is well to obtain certain useful relations among the current-amplitudes.

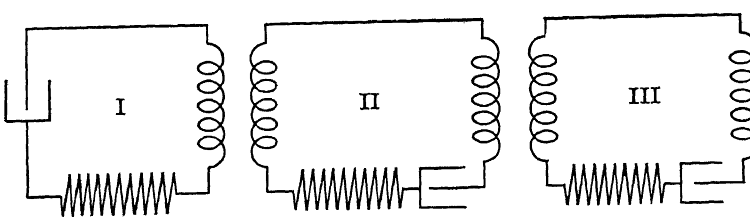

FIG. 3.—Chain of three circuits with transformer connections

189. Definition of a Chain of Circuits.—By a chain of circuits is meant a system in which the first circuit is coupled with the second, the second with the third, the third with the fourth, etc., to the last which is not connected to the first. That is, the chain is left open. Fig. 3 shows such a chain in which the connections are all through transformers. We may also have the connections or couplings made in any other way, as in Fig. 4, where some of the connections are by transformers, some by having

a common member of any kind of impedance, and some by a combination of transformer and common member.

190. Current-amplitude Relations in the Chain.—Let us suppose that we have an e.m.f. sinusoidal in character applied to the first circuit, and let us obtain expressions for the vector amplitude of current in each of the circuits of the chain. Using the exponential form of e.m.f. given in equation (5), we can write down a series of equations connecting the amplitudes with one another by using the general equations (21) into which we are to set equal to zero all of the complex mutual impedances m except those that have their subscripts a pair of consecutive numbers. This gives

$$\left.\begin{aligned} z_1A_1 - m_{12}A_2 &= E \\ -m_{21}A_1 + z_2A_2 - m_{23}A_3 &= 0 \\ -m_{32}A_2 + z_3A_3 - m_{34}A_4 &= 0 \\ -m_{43}A_3 + z_4A_4 - m_{45}A_5 &= 0 \\ -m_{54}A_4 + z_5A_5 &= 0 \end{aligned}\right\} \tag{29}$$

where it is supposed that we have five circuits in the chain.

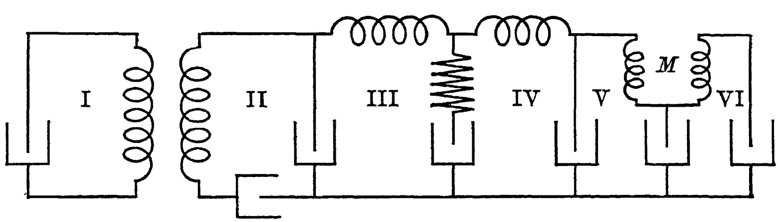

FIG. 4.—Chain of circuits with a variety of types of coupling.

These equations (29) may be solved by getting A_5 from the last equation, and substituting the result in the next preceding equation, etc., giving, in view of (19),

$$A_5 = \frac{m_{45}A_4}{z_5} \tag{30}$$

$$A_4 = \frac{m_{34}A_3}{z_4 - \dfrac{m_{45}^2}{z_5}} \tag{31}$$

$$A_3 = \frac{m_{23}A_2}{z_3 - \dfrac{m_{34}^2}{z_4 - \dfrac{m_{45}^2}{z_5}}} \tag{32}$$

$$A_2 = \cfrac{m_{12}A_1}{z_2 - \cfrac{m_{23}^2}{z_3 - \cfrac{m_{34}^2}{z_4 - \cfrac{m_{45}^2}{z_5}}}} \tag{33}$$

$$A_1 = \cfrac{E}{z_1 - \cfrac{m_{12}^2}{z_2 - \cfrac{m_{23}^2}{z_3 - \cfrac{m_{34}^2}{z_4 - \cfrac{m_{45}^2}{z_5}}}}} \tag{34}$$

The equations (30) *to* (34) *give the relations for finding the vector amplitudes of the currents in the several circuits of a chain with the coupling between the circuits of any character whatever. In this set of equations the e.m f. is applied to the first circuit and the chain is supposed to stop with the fifth circuit.*

If there are more than five circuits, it is evident from the form of the equations how the result may be extended to the greater number of circuits. If, on the other hand, there are fewer circuits than five, it is evident that all quantities having a subscript higher than the number of the circuits are to be set equal to zero.

It is also evident how the equations are to be changed in any case in which the e.m.f. is applied to Circuit V and the currents measured in the other circuits.

We shall next form a similar set of equations, when the e.m.f. is applied to some intermediate circuit.

In the equations (30) *to* (34), *the A's have values given by* (20) *and the m's by* (18).

191. Current-amplitudes When the E.M.F. is Applied to an Intermediate Circuit.—Let us suppose the e.m.f. to be applied to some intermediate circuit, say the *third* in the chain of circuits above referred to. In that case the equations (29) are the same as there given except that the amplitude E of e.m.f. is shifted from the first equation to the third. We may then get the relations among the current-amplitudes by starting with the last equations and successively eliminating up to the third, and then starting with the first and eliminating between successive equations down to the third, and by then solving the third equation. The result follows:

$$A_5 = \frac{m_{45}A_4}{z_5} \tag{35}$$

$$A_4 = \frac{m_{34}A_3}{z_4 - \dfrac{m_{45}^2}{z_5}} \tag{36}$$

$$A_3 = \frac{E}{z_3 - \dfrac{m_{23}^2}{z_2 - \dfrac{m_{12}^2}{z_1}} - \dfrac{m_{34}^2}{z_4 - \dfrac{m_{45}^2}{z_5}}} \tag{37}$$

$$A_2 = \frac{m_{23}A_3}{z_2 - \dfrac{m_{12}^2}{z_1}} \tag{38}$$

$$A_1 = \frac{m_{12}A_2}{z_1} \tag{39}$$

The equations (35) *to* (39) *give the relations for finding the vector current amplitudes when the e.m.f. is applied to the third circuit. The total number of circuits in the chain to which these equations apply is five. It will readily be seen how this result is to be modified for a different number of circuits, or for an application of e.m.f. to a different one of the intermediate circuits.*

The various equations (30) to (39) are given as models from which the vector current amplitudes may be obtained in a special case.

192. A Simplification is Introduced When m is Real or Pure Imaginary. Pure Mutual Impedance.—When the several m's are *real quantities or pure imaginaries,* a simplification is introduced in that all of the m^2's are reals. We can see under what conditions such a condition is attained, if we write down one of the m's in an expanded form. Take m_{12}, which expanded, becomes

$$m_{12} = R_{12} + j\left(L_{12}\omega - \frac{1}{C_{12}\omega} + M_{12}\omega\right) \tag{40}$$

If R_{12} alone enters, m_{12} is real; if R_{12} does not enter, m_{12} is a pure imaginary.

Some of the cases in which m_{12} is real or pure imaginary appear in the diagrams of Fig. 5.

The Circuits I and II themselves may have any character whatever, and there may be any number of them. The illustrations in Fig. 5 have reference only to the manner of coupling the circuits together. In the first diagram the two circuits are shown coupled together merely by having a common resistance, and in this case m_{12} is real. In all of the other diagrams of the figure m_{12} is shown as a pure imaginary.

When the coupling factor m_{12} is either real or pure imaginary we may appropriately call this factor *a pure mutual impedance,* to distinguish it from the general case of a *complex mutual impedance.* It is seen that the case of the pure *mutual impedance* covers many important systems of circuits, and we shall from here on confine the discussion to the systems of two or more circuits having pure *mutual impedances.*

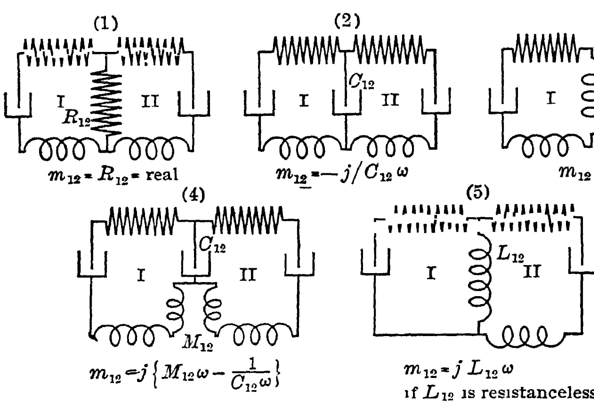

FIG 5.—Sample circuits with pure mutual impedance.

III. SOLUTION OF THE PROBLEM OF TWO CIRCUITS HAVING TRANSFORMER COUPLING

193. Statement of the Problem.—This is the problem of Chapters XI and XII. Given two circuits of the form shown in Fig.

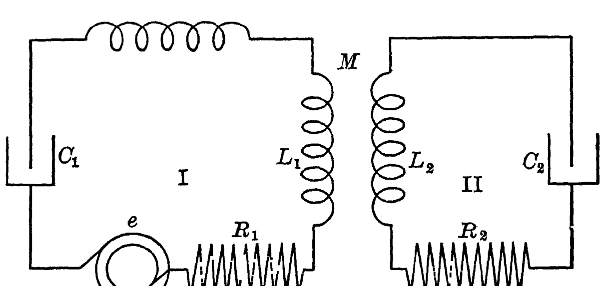

FIG. 6.—Two circuits with transformer coupling.

6, with a transformer connection between them, and with a sinusoidal e.m.f. impressed on Circuit I, to find the currents in the two circuits and to find also the resonance relations.

By (18) it is seen that the mutual impedance m_{12} is

$$m_{12} = jM\omega = m \text{ (say)} \tag{41}$$

where M is the mutual inductance between the circuits. This is a pure imaginary.

194. Currents When the E.M.F. is Applied to Circuit I.—By equations (30) to (34), using only the first two equations with changed subscripts, or using the last two equations with all terms of subscripts above 2 made zero, we have for the vector current amplitudes the equations

$$A_2 = \frac{mA_1}{z_2} \tag{41a}$$

$$A_1 = \frac{E}{z_1 - \dfrac{m^2}{z_2}} \tag{42}$$

where

$$\left.\begin{aligned} z_1 &= R_1 + jX_1 \\ z_2 &= R_2 + jX_2 \end{aligned}\right\} \tag{43}$$

As usual let

$$\left.\begin{aligned} Z_1^2 &= R_1^2 + X_1^2 \\ Z_2^2 &= R_2^2 + X_2^2 \end{aligned}\right\} \tag{44}$$

Replacing z_1 and z_2 in (41a) and (42) by their values and rationalizing the denominator of (42) we obtain

$$A_2 = \frac{mA_1}{R_2 + jX_2} \tag{45}$$

$$A_1 = \frac{E}{R'_1 + jX'_1} \tag{46}$$

where

$$R'_1 = R_1 - \frac{m^2R_2}{Z_2^2} \tag{47}$$

$$X'_1 = X_1 + \frac{m^2X_2}{Z_2^2} \tag{48}$$

Then by (8), using (45) and (46)

$$\dot{i}_1 = \frac{E}{R'_1 + jX'_1}e^{j\omega t} \tag{49}$$

$$\dot{i}_2 = \frac{m\dot{i}_1}{R_2 + jX_2} \tag{50}$$

From equation (49) it is seen that the current in Circuit I is the same as it would be if II were removed, and the resistance and reactance of Circuit I were replaced by R'_1 and X'_1 respectively. From (50) it is seen that the current in Circuit II is what it would be if I were removed and an e.m.f. mi_1 were impressed on Circuit II.

If now we replace m by its value (41), rationalize (49) and (50) and take the real part of the result we have

$$i_1 = \frac{E}{Z'_1} \cos\left(\omega t - \tan^{-1}\frac{X'_1}{R'_1}\right) \tag{51}$$

$$i_2 = \frac{M\omega E}{Z'_1 Z_2} \cos\left(\omega t - \tan^{-1}\frac{X'_1}{R'_1} + \frac{\pi}{2} - \tan^{-1}\frac{X_2}{R_2}\right) \tag{52}$$

Where

$$Z'_1 = \sqrt{X'^2_1 + R'^2_1} \tag{53}$$

R'_1, X'_1, Z'_1 are usually called *equivalent resistance, reactance and impedance* of the Circuit I. Since we are going to introduce certain other equivalences, we shall designate the equivalences here given *the forward equivalences.*

Equations (51) *and* (52) *give the current in the two circuits in a steady state when the cosine e.m.f. is impressed on Circuit I.*

195. Currents When the E.M.F. is Applied to Circuit II.— Let us next suppose that the e.m.f. is impressed, not on Circuit I, but on Circuit II, and let us call the equivalences in this case *backward equivalences,* which we shall indicate by an index (°). We can form the expressions for the current in this backward case by a mere interchange of subscripts 1 and 2 and an accompanying change of index from (′) to (°). That is,

With e.m.f. applied to Circuit II,

$$i_2 = \frac{E}{Z_2^\circ} \cos\left(\omega t - \tan^{-1}\frac{X_2^\circ}{R_2^\circ}\right) \tag{54}$$

$$i_1 = \frac{M\omega E}{Z_2^\circ Z_1} \cos\left(\omega t - \tan^{-1}\frac{X_2^\circ}{R_2^\circ} + \pi/2 - \tan^{-1}\frac{X_1}{R_1}\right) \tag{55}$$

There follows a table of equivalences (Table I) in which m is any pure mutual impedance. This Table I has application to any case of pure mutual impedance between the two circuits and may be used in a case more general than that of the transformer coupling here used in the illustration.

Table I.—Equivalences for two Circuits With Pure Mutual Impedance

Forward equivalences	Backward equivalences
$R'_1 = R_1 - \dfrac{m^2R_2}{Z_2^2}$	$R_2^\circ = R_2 - \dfrac{m^2R_1}{Z_1^2}$
$X'_1 = X_1 + \dfrac{m^2X_2}{Z_2^2}$	$X_2^\circ = X_2 + \dfrac{m^2X_1}{Z_1^2}$
$Z'^2_1 = R'^2_1 + X'^2_1$	$Z_2^{\circ 2} = R_2^{\circ 2} + X_2^{\circ 2}$

In the particular case under consideration, with transformer coupling,

$$m^2 = -M^2\omega^2 \tag{56}$$

We have in equations (51) *and* (52) *the current in the two circuits when the cosine e.m.f. is impressed on Circuit I; in equations* (54) *and* (55), *the corresponding currents when the cosine e.m.f. is applied to Circuit II. The Equivalences for two circuits with pure mutual impedance are given in Table I.*

196. Resonance Relations Obtained by the Theorem of Reciprocity.—We may now apply the Theorem of Reciprocity to determine the resonance relations in the system of two circuits with transformer coupling. The e.m.f. is to be applied to Circuit I, and we are to obtain the adjustment of either or both circuits such as to give a maximum of current amplitude in Circuit II. Calling the current amplitude in Circuit II I_2, we have, by (52)

$$I_2 = \frac{M\omega E}{Z'_1Z_2} \tag{57}$$

Now by the Reciprocity Theorem, this current amplitude is the same as the amplitude in Circuit I, with e.m.f. in Circuit II; that is, by (55) and the Reciprocity Theorem

$$I_2 = \frac{M\omega E}{Z_2^\circ Z_1} \tag{58}$$

The expressions (57) and (58) are now to be regarded merely as two different ways of writing I_2. In (57) Z'_1 is the only quantity that contains X_1, so the adjustment of X_1 that makes I_2 a maximum is that adjustment that makes Z'_1 a minimum; but since of the two terms that make up Z'_1, R'_1 does not contain X_1, we need only make X'^2_1 a minimum; and this is attained by making X'_1 zero. We have then that we obtain $X_{1\text{opt.}}$ by making

$$X'_1 = 0, \text{ for } X_{1\text{opt}} \tag{59}$$

In like manner, if we employ the second expression (58) for I_2, it is noticed that only $Z_2°$ contains X_2, and of this quantity $R_2°$ is independent of X_2, hence

$$X_2° = 0, \text{ for } X_{2_{\text{opt}}} \tag{60}$$

The result is this: For any given value of X_2, the optimum value of X_1 is that value that makes $X'_1 = 0$.

For any given value of X_1, the optimum value of X_2 is that value that makes $X_2° = 0$.

In order to get the grand maximum of current I_2 it is necessary to make both $X'_1 = 0$ and $X_2° = 0$.

197. Discussion of Results, and Their Reduction to the Forms Found in Chapter XI.—By Table I and equation (56), we may write equations (60) in the form

$$X_2 = \frac{M^2\omega^2X_1}{Z_1{}^2} \text{ gives } X_{2\text{ opt.}} \tag{61}$$

This equation is in agreement with (36) of Chapter XI, called *Partial Resonance Relation S*, and to it much of the discussion in Chapters XI and XII was devoted.

To get the current $I_{2\text{ max}}$ for X_2 optimum, it is only necessary to notice that in (58) of the present chapter, $Z_2°$ reduces to $R_2°$, so that (58) becomes in view of Table I

$$I_{2\text{ max.}} = \frac{M\omega E}{R_2Z_1 + \dfrac{M^2\omega^2R_1}{Z_1}}, \text{ at } X_{2\text{ opt.}} \tag{62}$$

This result agrees with (38) of Chapter X.

In order now to obtain the best adjustment of both circuits simultaneously, we may put (59) of the present chapter into the form

$$X_1 = \frac{M^2\omega^2X_2}{Z_2{}^2} \text{ gives } X_{1\text{ opt}} \tag{63}$$

and solve simultaneously with (61). Equation (63) agrees with equation (37) of Chapter XI, and was there called *Partial Resonance Relation P*.

As pointed out in Chapter XI one way of satisfying (61) and (63) simultaneously is by making

$$X_1 = 0 \text{ and } X_2 = 0 \tag{64}$$

Another way, by taking the ratio of (61) to (63), and applying the principle of *division* to the ratios, is by making

$$\frac{X_2}{X_1} = \frac{R_2}{R_1} = \frac{M^2\omega^2}{Z_1^2} \tag{65}$$

The equations (64) *and* (65) *are in agreement with equations* (43) *and* (45) *of Chapter XI, and give the optimum resonance relation.*

Now it is to be noticed, since Z_1^2 *is greater than or equal to* R_1^2, *that equation* (65) *can be fulfilled only provided*

$$R_1R_2 \gtreqless M^2\omega^2 \tag{66}$$

which is the criterion inequality (44) *of Chapter XI.*

The discussion of this problem will here be discontinued, because from this point forward the material beginning at equations

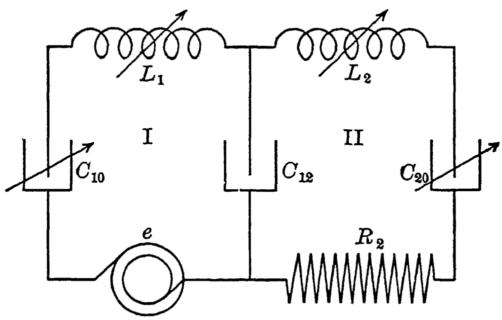

FIG. 7.—Two circuits with capacity coupling

(47) of Chapter XI and continuing through that chapter and through Chapter XII applies exactly.

IV. SOLUTION OF THE PROBLEM OF TWO CIRCUITS HAVING CAPACITY COUPLING

198. Statement.—Let us consider next two circuits coupled together by having a common condenser C_{12}, as shown in Fig. 7. The mutual impedance in this case is

$$m = 1/jC_{12}\omega \tag{67}$$

The reactances are

$$X_1 = L_1\omega - 1/C_1\omega, \quad X_2 = L_2\omega - 1/C_2\omega \tag{68}$$

where C_1 is total primary capacity consisting of C_{10} and C_{12} in

series, and C_2 is the total secondary capacity consisting of C_{20} and C_{12} in series.

Therefore,

$$\frac{1}{C_1} = \frac{1}{C_{10}} + \frac{1}{C_{12}}, \frac{1}{C_2} = \frac{1}{C_{20}} + \frac{1}{C_{12}} \tag{69}$$

The discussion concerning the case of the transformer coupling, given in the present chapter, applies exactly to the capacity coupling if we give to m the value in (67) in place of the value in (56).

With this understanding the table of equivalences Table I may be retained.

199. Current Amplitude in Circuit II.—The current-amplitude equations (57) and (58) must be changed by replacing $M\omega$ in the numerator[1] by $1/C_{12}\omega$. We thus obtain

$$I_2 = \frac{E}{C_{12}\omega Z'_1 Z_2} \tag{70}$$

and

$$I_2 = \frac{E}{C_{12}\omega Z_2{}^\circ Z_1} \tag{71}$$

Equations (70) *and* (71) *are alternative expressions for the current amplitude in Circuit II, when the coupling between Circuit I and Circuit II is by means of a common condenser* C_{12}. *The values of* Z'_1 *and* $Z_2{}^\circ$ *are given in Table I, which must be employed with the value of* m *given in* (67).

200. Partial Resonance Relations.—By replacing $M^2\omega^2$ by $1/C_{12}{}^2\omega^2$, equations (61) and (63) become the partial resonance relations for the capacity coupling, as follows:

$$X_2 = \frac{X_1}{C_{12}{}^2\omega^2 Z_1{}^2}, \text{ gives } X_{2\ \text{opt}} \tag{72}$$

and

$$X_1 = \frac{X_2}{C_{12}{}^2\omega^2 Z_2{}^2}, \text{ gives } X_{1\ \text{opt}} \tag{73}$$

In the case of capacity coupling by a condenser C_{12} *common to Circuit I and Circuit II, the value of* X_2 *given in* (72) *produces the largest current amplitude* I_2, *for given values of* C_{12}, X_1, Z_1, *and* ω.

[1] $\frac{1}{C_{12}\omega}$ with a minus sign is not employed because amplitude is essentially possible.

In like manner the value of X_1 given in (73) *produces the largest value of I_2, for given values of C_{12}, X_2, Z_2, and ω.*

201. Optimum Resonance Relation and Current at Optimum Resonance.—In order to obtain maximum current amplitude I_2 when both X_1 and X_2 are varied and adjusted, it is necessary to give to them such adjustments that both (72) and (73) are satisfied. Let us note that the product of (72) and (73) gives

$$X_1X_2 = \frac{X_1X_2}{C_{12}{}^4\omega^4Z_1{}^2Z_2{}^2}, \text{ whence (note also (72))} \tag{73}$$

Either
$$X_1 = 0 = X_2 \tag{74}$$
or
$$Z_1Z_2 = \frac{1}{C_{12}{}^2\omega^2} \tag{75}$$

The latter can be fulfilled only provided

$$\frac{1}{C_{12}{}^2\omega^2} \geqq R_1R_2$$

Returning to (72) and (73), let us divide one by the other obtaining

$$\frac{X_2{}^2}{X_1{}^2} = \frac{Z_2{}^2}{Z_1{}^2} \tag{76}$$

whence by *Division of Ratios*, and combination of results with (72) we have

$$\frac{X_2}{X_1} = \frac{R_2}{R_1} = \frac{1}{C_{12}{}^2\omega^2Z_1{}^2}, \text{ provided } \frac{1}{C_{12}{}^2\omega^2} > R_1R_2 \tag{77}$$

Also by (74)

$$X_1 = 0 = X_2, \text{ provided } \frac{1}{C_{12}{}^2\omega^2} < R_1R_2 \tag{78}$$

Equations (77) *and* (78) *are the optimum resonance relations. Out of analogy with the case of transformer coupling we may call* (77) *the optimum resonance relation with Capacity Coupling Sufficient, and* (78) *the optimum resonance relation with Capacity Coupling Deficient. Either relation is optimum when the Capacity Coupling is Critical (i.e., $1/C_{12}{}^2\omega^2 = R_1R_2$).*

202. Current Amplitude I_2 at Optimum Resonance. Max. Max. Current, with Capacity Coupling.—The resonance relation (72) is equivalent to making $X_2^\circ = 0$, as is seen by reference to Table I. In this case equation (71) becomes

$$[I_{2\,\text{max.}}]_{X_2\text{ opt.}} = \frac{E}{C_{12}\omega Z_1R_2^\circ} \text{ which, by Table I,}$$
$$= \frac{E}{C_{12}\omega\left\{Z_1R_2 + \dfrac{R_1}{C_{12}{}^2\omega^2Z_1}\right\}} \tag{79}$$

Equation (79) *gives the amplitude of current in Circuit II, when Circuit II has its optimum adjustment, with any values whatever of the other constants of the system.*

Let us now make the additional requirement that X_1 shall also have its optimum adjustment. There will be two cases according as the capacity coupling is Sufficient or Deficient.

First, with Coupling Sufficient, equation (77) gives

$$Z_1 = \frac{\sqrt{R_1}}{C_{12}\omega\sqrt{R_2}},$$

which introduced into (79) gives

$$I_{2\,\max\,\max} = \frac{E}{2\sqrt{R_1R_2}}, \text{ provided } \frac{1}{C_{12}{}^2\omega^2} > R_1R_2 \tag{80}$$

Second, with Coupling Deficient, we may still employ equation (79) but must satisfy (78) by making $X_1 = 0$. Then in (79) Z_1 reduces to R_1, and we obtain

$$I_{2\,\max\,\max} = \frac{E}{C_{12}\omega R_1R_2 + \frac{1}{C_{12}\omega}}, \text{ provided } \frac{1}{C_{12}{}^2\omega^2} < R_1R_2 \tag{81}$$

When Circuit I and Circuit II are coupled together by having a common condenser C_{12} we may designate the coupling as Sufficient when

$$\frac{1}{C_{12}{}^2\omega^2} > R_1R_2 \tag{82}$$

and may designate the Coupling as Deficient when

$$\frac{1}{C_{12}{}^2\omega^2} < R_1R_2 \tag{83}$$

When these two quantities that occur in (83) *are equal, we shall designate the Coupling as Critical. We have the result that* (80) *gives the max. max. current amplitude I_2 when the coupling is Capacity Coupling Sufficient. On the other hand, if the Capacity Coupling is Deficient, I_2 has the max. max. value given by* (81). *When the coupling is Critical, either* (80) *or* (81) *gives the current at optimum resonance of both Circuit I and Circuit II.*

As in the case of the transformer coupling, we have the result, that so long as the coupling is sufficient, the current I_2 at optimum adjustment has an amplitude independent of the size of the coupling-condenser C_{12}.

All of the deductions regarding the case of transformer coupling apply consistently to the case of capacity coupling, provided we replace $M\omega$ of the transformer case by $-1/C_{12}\omega$ of the capacity-coupling case.

V. SOLUTION OF THE PROBLEM OF TWO CIRCUITS HAVING RESISTANCE COUPLING

203. Partial Resonance Relations.—A diagram of two circuits coupled together by having a common resistance R_{12} is shown in Fig. 8. The common mutual impedance in this case is

$$m = R_{12} \tag{84}$$

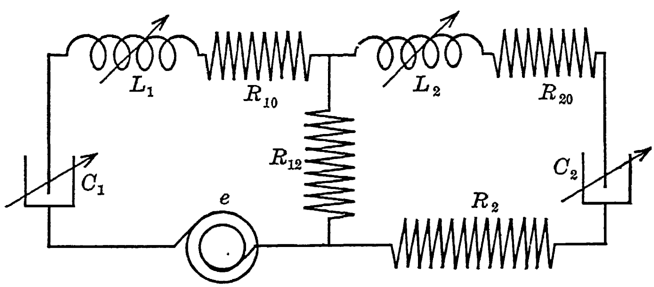

FIG. 8—Two circuits having resistance coupling.

The total resistance of the Circuit I is R_1 made up of the common resistance R_{12} and the resistance R_{10} (say) in Circuit I not common to Circuit II. That is,

$$R_1 = R_{10} + R_{12} \tag{85}$$

likewise

$$R_2 = R_{10} + R_{12} \tag{86}$$

With the understanding that m has the value given in (84) the Table of Equivalences (Table I) will give the Equivalences for this case also.

The partial resonance relations then become

$$X'_1 = 0, \text{ and } X_2^\circ = 0,$$

which by Table I and equation (84) give

$$X_1 = -\frac{R_{12}^2 X_2}{Z_2^2} \text{ gives } X_{1\text{opt}}. \tag{87}$$

and

$$X_2 = -\frac{R_{12}^2 X_1}{Z_1^2} \text{ gives } X_{2\text{ opt}}. \tag{88}$$

Equations (87) *and* (88) *are the partial resonance relations P and S respectively.*

204. Optimum Resonance Relation.—For the optimum resonance relation, (87) and (88) must be true simultaneously. They can be true simultaneously only provided

$$X_1 = 0 = X_2 \tag{89}$$

for by taking the product of (87) and (88) we have

$$X_1X_2(Z_1^2Z_2^2) = X_1X_2(R_{12}^4) \tag{90}$$

Now $Z_1Z_2 \geqq R_1R_2$, by definition of Z_1 and Z_2, and by (85) and (86) $R_1R_2 > R_{12}^2$, hence

$$Z_1^2Z_2^2 > R_{12}^4,$$

and by (90), therefore

$$X_1X_2 = 0.$$

Comparison of this result with (87) and (88) shows that both X_1 and X_2 must be zero.

In this case of resistance coupling between the circuits I and II, we have only one case of optimum resonance, given by (89), *which corresponds to the case of Deficient Coupling in the other examples of Transformer Coupling and Capacity Coupling.*

205. Secondary Current Amplitude at Optimum Resonance with Resistance Coupling.—The general expression for amplitude I_2 of current in this case, since this amplitude is essentially positive, is obtained by replacing $M\omega$ by R_{12} in equations (57), and is

$$I_2 = \frac{R_{12}E}{Z'_1Z_2} = \frac{R_{12}E}{Z_2^\circ Z_1}.$$

Before passing to the case of optimum resonance, let us introduce merely the resonance relation with X_2 optimum as given in (88), which is equivalent to $X_2^\circ = 0$. This gives

$$[I_{2\,\text{max.}}]_{X_2\,\text{opt}} = \frac{R_{12}E}{Z_1R_2^\circ}$$

and, by Table I,

$$= \frac{R_{12}E}{Z_1\left(R_2 - \dfrac{R_{12}^2R_1}{Z_1^2}\right)} \tag{91}$$

Equation (91) *gives the amplitude of current in Circuit II when all the constants, except* X_2, *have any values, and* X_2 *has its optimum adjustment as specified by* (88).

Let us now introduce the condition that X_1 as well as X_2 shall have its optimum adjustment. By (89) this can be attained only by making $X_1 = 0$, then by (88) X_2 automatically becomes zero.

Making $X_1 = 0$ in (91) we obtain

$$I_{2\,\text{max. max}} = \frac{R_{12}E}{R_1R_2 - R_{12}^2} \tag{92}$$

Equation (92) *gives the maximum possible value of* I_2, *in the case of two circuits I and II coupled by having a common resistance* R_{12}. *The adjustment that gives this max. max. current is given by* (89), *and is seen to be an adjustment of each circuit separately to have its undamped period equal to the period of the impressed e.m.f.*

Note that the case of resistance coupling is always one of essentially Deficient Coupling.

CHAPTER XIV

RESONANCE RELATIONS IN A CHAIN OF THREE CIRCUITS WITH CONSTANT PURE MUTUAL IMPEDANCES. STEADY STATE

206. Statement of Problem.—We propose now to utilize the Reciprocity Theorem of the preceding chapter to determine the resonance relations in a system of three circuits arranged in a chain with the couplings between the circuits in the form of *pure mutual impedances*, as defined in Art. **192.** The purpose of this treatment is, first, to give an illustration of the simplicity resulting from the use of the Reciprocity Theorem to determine resonance relations, and, second, to lay the foundations for solving important problems relating to radiotelegraphic practice.

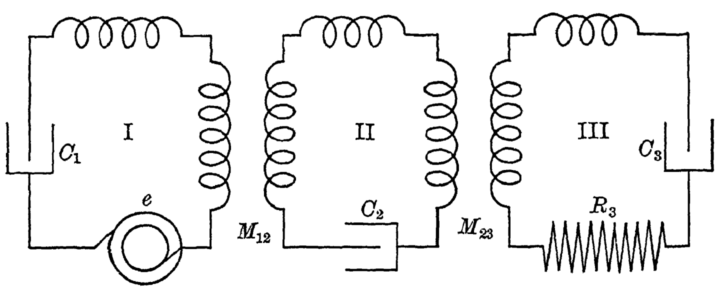

FIG. 1—Chain of three circuits with transformer coupling.

207. Illustrative Forms of Circuits.—Two forms of circuits to which the present analysis applies are shown in Figs. 1 and 2.

In Fig. 1 the couplings in the chain of three circuits are made by transformers.

In Fig. 2, which is analogous to a much-used type of radio receiving system, the coupling between Circuit I and Circuit II is by a transformer, while the coupling between Circuits II and III is by a common condenser C_{23}.

In both figures R_3 represents a resistance that may be regarded as the resistance of the detector.

The two figures are both special cases of a chain of three cir-

cuits with *pure mutual impedances* as defined in the previous chapter.

208. Anticipatory Sketch of the Method.—The method employed in this problem will consist in obtaining three Variant Expressions for the current in Circuit III, when the e.m.f. is applied to Circuit I. These three forms will be found to be

$$I_3 = \underset{(1)}{\frac{|\beta\gamma E|}{Z_3Z'_2Z'_1}} = \underset{(2)}{\frac{|\beta\gamma E|}{Z_1Z_2{}^\circ Z_3{}^\circ}} = \underset{(3)}{\frac{|\beta\gamma E|}{Z_1Z'_2{}^\circ Z_3}}$$

where the vertical lines enclosing the numerator *indicates absolute value.*

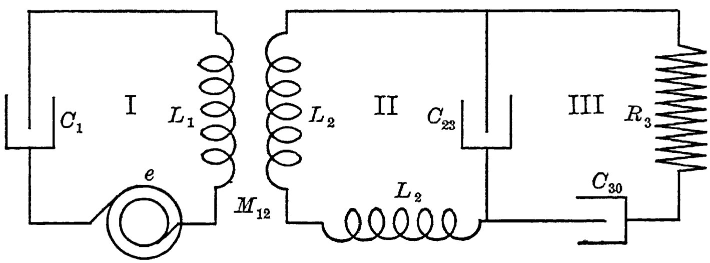

FIG. 2.—Chain of three circuits with one transformer coupling and one capacity coupling.

The various Z's will be found to have the definitions given in Table I, Art. **211**. The values of the various Z's will then be shown to be such that we obtain certain fundamental forms of the resonance relations by inspection.

The principles underlying the method will now be established, first, by directly showing the identity of the denominators of (1), (2), and (3), and, second, by the use of the Reciprocity Theorem.

209. Direct Proof of the Identity of the Denominators of Equations (1), (2), and (3).—Referring to Table I in Art. **211** for definitions of the various Z's, let us note by direct multiplication and substitution that

$$\left.\begin{aligned}(Z_3Z'_2Z'_1)^2 &= (Z_1Z_2{}^\circ Z_3{}^\circ)^2 = (Z_1Z'_2{}^\circ Z_3)^2\\ &= Z_1{}^2Z_2{}^2Z_3{}^2 + \beta^4Z_3{}^2 + \gamma^4Z_1{}^2\\ &+ 2\beta^2Z_3{}^2(R_1R_2 - X_1X_2)\\ &+ 2\gamma^2Z_1{}^2(R_2R_3 - X_2X_3)\\ &+ 2\beta^2\gamma^2(R_1R_3 + X_1X_3)\end{aligned}\right\} \qquad (4)$$

From equations (4) it appears that equations (1), (2), and (3) are established as soon as we prove the correctness of any one of

them. This last step is easy to take, but will be here omitted, as the step occurs in the use of the Reciprocity Theorem following.

On account of the importance of the Reciprocity Theorem in itself, we shall now make use of it to deduce again the identity of equations (1), (2), and (3), and shall incidentally supply such steps as have been omitted in the above sketch.

I. APPLICATION OF THE RECIPROCITY THEOREM

210. Notation.—The notation employed here will be the same as in the preceding chapter, namely, as we go around the nth circuit,

R_n = the sum of all the resistances in series in the nth circuit including resistances common to neighboring circuits, if there be such;

L_n = the sum of all self-inductances in series in the nth circuit, including self-inductances common to the nth circuit and its neighbors if there be such, and including the self-inductance of any primary or secondary coil of a transformer if any such coil be in the nth circuit;

$1/C_n$ = the sum of the reciprocals of all capacities in series in the nth circuit, including the capacities of condeners; common to the nth circuit and to neighboring circusits

$X_n = L_n\omega - 1/C_n\omega,\quad Z_n^2 = R_n^2 + X_n^2,\quad z_n = R_n + jX_n;$

m_{12} = complex mutual impedances between Circuits I and II $= z_{12} + jM_{12}\omega$, where

z_{12} = complex impedance common to Circuits I and II, if there be such, and

M_{12} = mutual inductance between Circuits I and II, if there be such.

m_{23}, m_{34}, etc. = similar quantity to m_{12} but for other pairs of circuits.

211. Values of Complex Current Amplitudes and Complex Currents.—By means of the general methods of Chapter XIII, it is seen that with the cosine e.m.f. applied to Circuit I, the currents in the three circuits are the *real parts of the complex quantities*

$$i_1 = A_1\epsilon^{j\omega t},\quad i_2 = A_2\epsilon^{j\omega t},\quad i_3 = A_3\epsilon^{j\omega t} \tag{5}$$

where ω is the angular velocity of the impressed e.m.f., and A_1, A_2, A_3, satisfy relations of the form of (34), (33) and (32) of

Chapter XIII, with, however, all of the terms of subscripts higher than 3 made equal to zero. These relations written out here are

$$A_3 = \frac{m_{23}A_2}{z_3} \qquad = \frac{m_{23}A_2}{R_3 + jX_3} \tag{6}$$

$$A_2 = \frac{m_{12}A_1}{z_2 - \dfrac{m_{23}^2}{z_3}} \qquad = \frac{m_{12}A_1}{R'_2 + jX'_2} \tag{7}$$

$$A_1 = \frac{E}{z_1 - \dfrac{m_{12}^2}{z_2 - \dfrac{m_{23}^3}{z_3}}} = \frac{E}{R_1' + jX'_1} \tag{8}$$

where the third members of (7) and (8) are written down from the general knowledge that any algebraic combination of complex quantities is a complex quantity of the form $a + jb$.

Equations (7) and (8) require that R'_1, X'_1, R'_2, and X'_2 shall be given definitions consistent with these equations (7) and (8).

By working out the values of the denominators in (7) and (8), and equating the two denominators for the same quantity in each case, we obtain the values of the primed quantities in the first column of Table I following:

Table I.—Equivalences for Three Circuits with Pure Mutual Impedances

Forward equivalences	Backward equivalences	Two-way equivalences
$R'_1 = R_1 + \dfrac{\beta^2 R'_2}{Z'^2_2}$	$R_3^\circ = R_3 + \dfrac{\gamma^2 R_2^\circ}{Z_2^{\circ 2}}$	
$X'_1 = X_1 - \dfrac{\beta^2 X'_2}{Z'^2_2}$	$X_3^\circ = X_3 - \dfrac{\gamma^2 X_2^\circ}{Z_2^{\circ 2}}$	
$R'_2 = R_2 + \dfrac{\gamma^2 R_3}{Z_3^2}$	$R_2^\circ = R_2 + \dfrac{\beta^2 R_1}{Z_1^2}$	$R'^\circ_2 = R_2 + \dfrac{\beta^2 R_1}{Z_1^2} + \dfrac{\gamma^2 R_3}{Z_3^2}$
$X'_2 = X_2 - \dfrac{\gamma^2 X_3}{Z_3^2}$	$X_2^\circ = X_2 - \dfrac{\beta^2 X_1}{Z_1^2}$	$X'^\circ_2 = X_2 - \dfrac{\beta^2 X_1}{Z_1^2} - \dfrac{\gamma^2 X_3}{Z_3^2}$
$Z'^2_1 = R'^2_1 + X'^2_1$	$Z_3^{\circ 2} = R_3^{\circ 2} + X_3^{\circ 2}$	$Z'^{\circ 2}_2 = R'^{\circ 2}_2 + X'^{\circ 2}_2$
$Z'^2_2 = R'^2_2 + X'^2_2$	$Z_2^{\circ 2} = R_2^{\circ 2} + X_2^{\circ 2}$	

In Table I and in subsequent equations, since the two m's are pure imaginaries, as may be seen by reference to their formation, we have let

$$\left.\begin{aligned} m_{12} &= j\beta \\ \text{and} \qquad m_{23} &= j\gamma \end{aligned}\right\} \tag{9}$$

where β and γ are real quantities.

In setting up Table I we have replaced m_{12} *and* m_{23} *by their values* (9).

212. Currents in Terms of Forward Equivalences.—We may now write down the values of the currents i_1, i_2, i_3 with the use of the Forward Equivalences contained in column one of Table I.

This is done by taking (6), (7), and (8), in terms of the primed quantities, eliminating among them and substituting the results in (5), and then rationalizing and taking the real part of the result, obtaining

$$i_1 = \frac{E}{Z'_1} \cos (\omega t - \varphi'_1) \tag{9}$$

$$i_2 = \frac{\beta E}{Z'_2 Z'_1} \cos (\omega t + \pi/2 - \varphi'_1 - \varphi'_2) \tag{10}$$

$$i_3 = \frac{\beta\gamma E}{Z_3 Z'_2 Z'_1} \cos (\omega t + \pi - \varphi'_1 - \varphi'_2 - \varphi_3) \tag{11}$$

where

$$\varphi'_1 = \tan^{-1} \frac{X'_1}{R'_1},\ \varphi'_2 = \tan^{-1} \frac{X'_2}{R'_2},\ \varphi_3 = \tan^{-1} \frac{X_3}{R_3} \tag{12}$$

Also, if in (11) we let I_3 be the amplitude of i_3, we have

$$I_3 = \frac{\beta\gamma E}{Z_3 Z'_2 Z'_1} \tag{13}$$

Equations (9), (10), *and* (11) *give the values of the currents in the three circuits respectively after these currents have reached a steady state, under the action of a cosine e.m.f. of amplitude E impressed on Circuit I. Equation* (13) *gives the amplitude of current in Circuit III.*

213. Current Amplitude I_3 in Terms of Backward Equivalences. Let us now obtain a Variant form of I_3. To do this we shall temporarily suppose that the e.m.f. is applied to Circuit III, and shall obtain the current in Circuit I. The Backward Equivalences of Table I bear to this case the same relation that the Forward Equivalences bear to the forward case, so we obtain

$$I_1 = \frac{\beta\gamma E}{Z_1 Z_2^\circ Z_3^\circ}, \text{ when the e.m.f. is applied to Circuit III.}$$

Now by the Reciprocity Theorem, this current I_1 is the same as we should get in Circuit III (that is, I_3) if the e.m.f. were applied to Circuit I; whence

$$I_3 = \frac{\beta\gamma E}{Z_1 Z_2^\circ Z_3^\circ}, \text{ with e.m.f. in Circuit I.}$$

This is equation (2) above.

As to equations (1), let us note that it has been already obtained in (13).

214. Current Amplitude I_3 in Terms of Two-way Equivalences. We have, remaining, one more form of expression (3) to obtain for I_1. This may be obtained by the Theorem of Reciprocity applied to Circuits I and II. By the general equations of the form of (29), Chapter XIII, when the e.m.f. is applied to Circuit II, and when there are only three circuits in the chain, we obtain the relations

$$\left.\begin{aligned} z_1A_1 - m_{12}A_2 &= 0 \\ -m_{12}A_1 + z_2A_2 - m_{12}A_3 &= E \\ -m_{23}A_2 + z_3A_3 \qquad &= 0 \end{aligned}\right\} \tag{13}$$

Replacing m_{12} by $j\beta$, m_{23} by $j\gamma$, and solving (13) for A_1, A_2, and A_3, we obtain

$$\left.\begin{aligned} A_1 &= \frac{j\beta A_2}{z_1} \\ A_3 &= \frac{j\gamma A_2}{z_3} \\ A_2 &= \frac{E}{z_2 + \dfrac{\beta^2}{z_1} + \dfrac{\gamma^2}{z_3}} = \frac{E}{R'^{\circ}_2 + jX'^{\circ}_2} \text{ (say)} \end{aligned}\right\} \tag{14}$$

The last denominator of the A_2-equation is an abbreviation for the complex denominator preceding it in the A_2-equation. Equating the real and the imaginary parts of these two denominators respectively, we obtain, on solving, the values of R'°_2 and X'°_2 contained in the last column of Table I. These values are the equivalent resistance and reactance of Circuit II as influenced by the two Circuits I and II, and are hence called the *Two-way Equivalences* of Circuit II.

Now solving the A_1-equation and the A_2-equation of (14) as simultaneous, rationalizing and taking the amplitude of the real part, we obtain

$$I_1 = \frac{\beta E}{Z'^{\circ}_2 Z_1}, \text{ with e.m.f. in II} \tag{15}$$

Compare with this the amplitude of i_2 in (10), which gives

$$I_2 = \frac{\beta E}{Z'_1 Z'_2}, \text{ with e.m.f in I} \tag{16}$$

By the Theorem of Reciprocity these two quantities (15) and (16) are equal, hence

$$Z'_2{}^{\circ}Z_1 = Z'_1 Z'_2 \tag{17}$$

Multiplying both sides of this equation by Z_3, we obtain

$$Z_1 Z'_2{}^{\circ} Z_3 = Z_3 Z'_2 Z'_1 \tag{18}$$

which makes (3) true if (1) is true. But we have already proved (1).

We have thus shown that equations (1), (2), *and* (3) *are three different ways of expressing the current amplitude in Circuit III under the action of a cosine e.m.f. applied to Circuit I, provided the current has reached a practically steady state.*

The use of the Reciprocity Theorem has enabled us to obtain certain Equivalent Resistances, indicated by R'_1, R'_2, $R_3{}^{\circ}$, $R_2{}^{\circ}$, $R'_2{}^{\circ}$, *and certain Equivalent Reactances, indicated by* X'_1, X'_2, $X_3{}^{\circ}$, $X_2{}^{\circ}$, $X'_2{}^{\circ}$, *all of which are tabulated with their values in Table I. By taking the square root of the sum of the squares of these resistances and the corresponding reactances we have formed, and included in Table I, the Equivalent Impedances* Z'_1, Z'_2, $Z_3{}^{\circ}$, $Z_2{}^{\circ}$, $Z'_2{}^{\circ}$.

We have then written down in terms of the Equivalent Impedances three different expressions for the Current Amplitude I_3. *In these three expressions* (1), (2), (3), *the occurrence of* X_1, X_2, *and* X_3, *as will presently be shown, is such that certain fundamental forms of the resonance relations may be had by inspection.*

II. PARTIAL RESONANCE RELATIONS AND RESTRICTED RESONANCE RELATIONS WITH PURE MUTUAL IMPEDANCES UNCHANGED

215. Nomenclature.—We shall designate as *Partial Resonance Relation re* X_1 the adjustment of X_1 that makes I_3 (say) a maximum when all the other members of the circuits are kept constant.

In general a *Partial Resonance Relation re a Variable* will mean the adjustment of the variable that makes the amplitude of the current in the detector circuit (or work circuit) a maximum while all the other members of the system are kept constant.

In certain cases the range of adjustment of a designated variable may not be sufficient to attain an absolute maximum

of the work current. In those cases we shall designate as a *Restricted Resonance Relation re a Variable* the adjustment of the variable that will make the current amplitude in the work circuit the largest that can be obtained with any adjustment possible to the variable under the limitations of the restriction. In case, for example, X_1 is the variable under observation, we shall refer to the value of X_1 that gives the greatest work current, subject to the restrictions of X_1, as the *Restricted Resonance Relation re* X_1, *or Resonance Relation re* X_1 *Restricted.*

216. Resonance Relations for a Chain of Three Circuits With Pure Mutual Impedances Unchanged.—We have already pointed out in the anticipatory sketch (Art. **208**) the nature of the steps to be employed. Three forms of expression for I_3 were given in equations (1), (2), and (3), and these three forms have now been derived and shown to be identical in value. Since the numerators are supposed to be constant, we can make I_3 a maximum, by making the denominators a minimum.

By definition of the various equivalences in Table I it is seen that the denominator $Z_3Z'_2Z'_1$, of equations (1) involves X_1 only in the factor Z'_1. To make I_3 a maximum by varying X_1, it is necessary, therefore, only to make Z'_1 a minimum re X_1. Since the resistances of the system are all constants, in it is seen, by reference to Table II, that this is attained by making X'^2_1 a minimum re X_1

Hence, if X_1 is unrestricted, the resonance condition is

$$X'_1 = 0 \tag{20}$$

(Partial Resonance Relation re X_1)

On the other hand, if X_1 is restricted, the resonance condition is

$$X'^2_1 = \text{minimum} \tag{21}$$

(Resonance Relation re X_1 Restricted)

In like manner, since in the denominator of (3), X_2 occurs only in the factor Z'°_2, we find, by similar reasoning,

$$X'^\circ_2 = 0 \tag{22}$$

(Partial Resonance Relation re X_2)

and

$$X'^{\circ 2}_2 = \text{minimum} \tag{23}$$

(Resonance Relation re X_2 Restricted)

Again, since in the denominator of (2) X_3 occurs only in the factor Z_3°, we have

$$X_3^\circ = 0 \tag{24}$$

(Partial Resonance Relation re X_3)

and

$$X_3^{\circ 2} = \text{minimum} \tag{25}$$

(Resonance Relation re X_3 Restricted)

Equations (20), (22), *and* (24) *give respectively the partial resonance relations re* X_1, X_2, *and* X_3, *when the mutual impedances are pure and unvaried. In case restrictions on any or all of the reactances prohibits the attainment of any or all of the partial resonance relations, we must substitute for any of the relations that is unattainable the corresponding Resonance Relation Restricted, as given in* (21), (23), *or* (25).

III. APPLICATION TO A CASE IN WHICH THE REACTANCES ARE ALL UNRESTRICTED

217. Optimum Resonance Adjustments. Adjustments for a Grand Maximum of Current Amplitude I$_3$, When the Reactances are All Unrestricted.—Let us now determine the adjustments that must be given to all three of the circuits, in order to obtain a grand maximum of amplitude I_3, under the condition that all of the reactances are unrestricted.

This is done by solving (20), (22), and (24) as simultaneous. For this purpose we shall make constant use of Table I, Art. **211.**

Let us first solve (20) and (22) as simultaneous.

By (22)

$$X'^\circ_2 = 0.$$

By a comparison of the third and first columns of Table I, Art. **211,** it is seen that the satisfaction of this equation requires

$$X'_2 - \frac{\beta^2 X_1}{Z_1^2} = 0 \tag{26}$$

Equation (26) *is an alternative form of* (22).

Returning now to (20), we may write it (by the definition of X'_1) in the form

$$X_1 - \frac{\beta^2 X'_2}{Z'^2_2} = 0 \tag{27}$$

Equation (27) *is an alternative form of* (20).

If now (22) and (20) are simultaneously true, their equivalents (26) and (27) must be simultaneously true; so that by replacing X'_2 in the numerator and denominator of (27) by its value from (26), we obtain

$$X_1 = \frac{\beta^4 X_1}{Z_1^2 \left(R'^2_2 + \frac{\beta^4 X_1^2}{Z_1^4}\right)} \tag{28}$$

Equation (28) *is a first step in the treatment of* (20) *and* (22) *as simultaneous.*

From (28) it follows that

either $$X_1 = 0 \tag{29}$$

or $$R'^2_2 = \frac{\beta^4}{Z_1^2} - \frac{\beta^4 X_1^2}{Z_1^4} = \frac{\beta^4 R_1^2}{Z_1^4} \tag{30}$$

This last equation is obtained by dividing (28) by X_1, and clearing of fractions, obtaining the equality of the first term to the second. The third member follows from the second by employing the definition of Z_1^2.

Extracting the square root of (30) and combining the alternative combination (29) (30) with (26), we obtain

either $$X_1 = 0 \text{ and } X'_2 = 0 \tag{31}$$

or $$\frac{X'_2}{X_1} = \frac{R'_2}{R_1} = \frac{\beta^2}{Z_1^2} \tag{32}$$

Equations (31) *and* (32) *constitute a pair of results, one or the other of which must be fulfilled in order to make X_1 and X_2 both optimum, while X_3 may have any value whatever. The quantity X_3 is involved in X'_2 and R'_2 (see Table I).*

A similar treatment of (22) and (24) as simultaneous gives

either $$X_3 = 0 \text{ and } X_2^\circ = 0 \tag{33}$$

or $$\frac{X_2^\circ}{X_3} = \frac{R_2^\circ}{R_3} = \frac{\gamma^2}{Z_3^2} \tag{34}$$

Equations (33) *and* (34) *constitute a pair of results, one or the other of which must be fulfilled in order to make X_3 and X_2 both optimum, while X_1 (involved in X_2° and R_2°) may have any values whatever.*

We come next to treat of the case where all three of the circuits are at optimum adjustment simultaneously. This treatment consists in solving the equations (31) and (32) as

simultaneous with (33) and (34), while keeping in mind that the two pairs of equations are themselves alternative possibilities.

We shall first show that (32) and (34) are *not* simultaneously possible, as follows:

Replacing the primed quantities in (32) by their values from Table I, we obtain for this equation

$$\frac{X_2}{X_1} - \frac{\gamma^2 X_3}{X_1 Z_3^2} = \frac{R_2}{R_1} + \frac{\gamma^2 R_3}{R_1 Z_3^2} = \frac{\beta^2}{Z_1^2} \tag{35}$$

A similar treatment of (34) gives for it

$$\frac{X_2}{X_3} - \frac{\beta^2 X_1}{X_3 Z_1^2} = \frac{R_2}{R_3} + \frac{\beta^2 R_1}{R_3 Z_1^2} = \frac{\gamma^2}{Z_3^2} \tag{36}$$

If we make these two results true simultaneously their latter parts lead to

$$2R_2/R_3 = 0,$$

which cannot be true.

We may, therefore, exclude the simultaneous fulfillment of (32) *and* (34) *as a possible compliance with the resonance requirement.*

We shall next examine (31) and (33) as a possible simultaneous resonance adjustment. This combination gives

$$X_1 = 0, X_3 = 0, X'_2 = 0, X_2^\circ = 0$$

By definitions of X'_2 and X_2° (Table I), these equations reduce to

$$X_1 = X_2 = X_3 = 0 \tag{37}$$

Equations (37) *is the result of treating* (31) *and* (33) *as simultaneous.*

Let us now examine the combination of (31) and (34). By (31) two of the numerator terms of (34) reduce to simpler values, and the combination gives

$$X_1 = 0, \text{ and } \frac{X_2}{X_3} = \frac{R_2 R_1 + \beta^2}{R_1 R_3} = \frac{\gamma^2}{Z_3^2} \tag{38}$$

Equations (38) *is the result of treating* (31) *and* (34) *as simultaneous.*

In like manner the combination of (33) and (32) gives

$$X_3 = 0, \text{ and } \frac{X_2}{X_1} = \frac{R_2 R_3 + \gamma^2}{R_1 R_3} = \frac{\beta^2}{Z_1^2} \tag{39}$$

Equations (39) *is the result of treating* (33) *and* (32) *as simultaneous.*

218. Adjustments for Grand Maxima of I_3 Summarized and Designated Optimum Combinations. Current Amplitude I_3 Obtained at the Optimum Combinations. Conditions Under Which the Combinations are Respectively Optimum.—We have given in equations (37), (38), and (39) three combinations of relations any one of which satisfies (20), (22), and (24) simultaneously, and is a possible optimum combination. We shall now show that it is sometimes one and sometimes another of these combinations that is optimum.

Let us designate the three combinations as follows:

Optimum Combination (α), equation (37);
Optimum Combination (β), equation (38);
Optimum Combination (γ), equation (39).

The condition under which Combination (β) is attainable may be had by inspection of (38), by noting that Z_3^2 cannot be less than R_3^2, whence

$$\frac{R_1R_2 + \beta^2}{R_1R_3} \gtreqless \frac{\gamma^2}{R_3^2};$$

that is,

$$R_1R_2R_3 \gtreqless R_1\gamma^2 - R_3\beta^2. \tag{40}$$

The inequality 40 *gives the condition under which Combination* (β) *is attainable.*

A similar process shows the condition under which (γ) is attainable, and gives

$$R_1R_2R_3 \gtreqless R_3\beta^2 - R_1\gamma^2 \tag{41}$$

The inequality (41) *gives the condition under which Combination* (γ) *is attainable.*

There is no restriction on the attainability of Combination (α).

To find the current amplitude I_3 under the three Optimum Combinations respectively, let us take I_3 in the form given in equation (3). This is

$$I_3 = \frac{[\beta\gamma E]}{Z_1Z_3Z'_2}\cdot \tag{42}$$

On substituting the combination of equations (37) into this, we have for the value of I_3, under the optimum combination (α), the value

$$[I_3]_\alpha = \frac{[\beta\gamma E]}{R_1R_2R_3 + \beta^2R_3 + \gamma^2R_1} \tag{43}$$

Likewise in (42) substituting the optimum combination (β) as given by (38), we obtain after reduction

$$[I_3]_\beta = \frac{[\beta E]}{2\sqrt{R_1R_3}\,\sqrt{R_1R_2 + \beta^2}} \tag{44}$$

Again in (42) substituting the optimum combination (γ) as given by (39), we obtain after reduction.

$$[I_3]_\gamma = \frac{[\gamma E]}{2\sqrt{R_1R_3}\,\sqrt{R_3R_2 + \gamma^2}} \tag{45}$$

It will now be shown that $[I_3]_\beta$, whenever (β) is attainable is larger than $[I_3]_\alpha$. This is done by multiplying the numerator and denominator of (44) by γ, which makes the numerator the same as the numerator of (43). A comparison of the resultant denominators now shows that

$$[I_3]_\beta \gtreqless [I_3]_\alpha$$

whenever

$$R_1R_2R_3 + \beta^2R_3 + \gamma^2R_1 - 2\gamma\sqrt{R_1R_3}\,\sqrt{R_1R_2 + \beta^2} \gtreqless 0 \tag{46}$$

The left-hand side may be expressed as a square thus

$$\left\{\sqrt{R_3}\sqrt{R_1R_3 + \beta^2} - \gamma\sqrt{R_1}\right\}^2 \gtreqless 0,$$

which is seen to be always fulfilled, since the quantities under the radicals are all positive.

We have then the result that Combination (β), if attainable, gives a larger value of I_3 than does Combination (α). In a similar way it can be proved that Combination (γ), if attainable also beats (α). It is not necessary to compare (β) with Combination (γ) since the two are never both attainable in the same case, as may be seen by comparing (40) with (41).

The results may now be further summarized in the following Key.

219. Summary and Key Concerning Grand Maxima of I_3 When the Mutual Impedances are Invariable, and When the Reactances are Unrestricted.—

I. Resonance Combination (α).

If

$$R_1R_2R_3 \gtreqless [R_1\gamma^2 - R_3\beta^2],$$

where the vertical lines indicate "absolute value," use Resonance Relation

$$X_1 = X_2 = X_3 = 0 \tag{47}$$

and calculate the grand maximum of I_3 by

$$[I_3]_\alpha = \frac{[\beta\gamma E]}{R_1R_2R_3 + \beta^2R_3 + \gamma^2R_1} \tag{48}$$

II. Resonance Combination (β).
If

$$R_1R_2R_3 \gtreqless R_1\gamma^2 - R_3\beta^2,$$

use Resonance Relations

$$X_1 = 0, \frac{X_2}{X_3} = \frac{R_1R_2 + \beta^2}{R_1R_3} = \frac{\gamma^2}{Z_3^2} \tag{49}$$

and calculate the grand maximum of I_3 by

$$[I_3]_\beta = \frac{[\beta E]}{2\sqrt{R_1R_3}\ \sqrt{R_1R_2 + \beta^2}} \tag{50}$$

III. Resonance Combination (γ).
If

$$R_1R_2R_3 \gtreqless R_3\beta^2 - R_1\gamma^2,$$

use Resonance Relations

$$X_3 = 0, \frac{X_2}{X_1} = \frac{R_2R_3 + \gamma^2}{R_1R_3} = \frac{\beta^2}{Z_1^2} \tag{51}$$

and calculate the grand maximum of I_3 by

$$[I_3]_\gamma = \frac{[\gamma E]}{2\sqrt{R_1R_3}\ \sqrt{R_3R_2 + \gamma^2}} \tag{52}$$

This summary, or key, contains the optimum resonance combinations and the grand maxima of current in Circuit III, obtained when the reactances X_1, X_2, and X_3 are unrestricted.

CHAPTER XV

RESONANCE RELATIONS IN A RADIOTELEGRAPHIC RECEIVING STATION HAVING A COUPLED SYSTEM OF CIRCUITS WITH THE DETECTOR IN SHUNT TO A SECONDARY CONDENSER

I. GENERAL RESULTS

220. Form of Circuits.—In Chapters XI and XII there is given a theory of coupled circuits approximately applicable to a radiotelegraphic receiving station in which the detector is in series in the secondary circuit. The treatment is approximate in that the receiving antenna of practice had its capacity, inductance, and resistance distributed along the length of the antenna, while the system treated was idealized by replacing

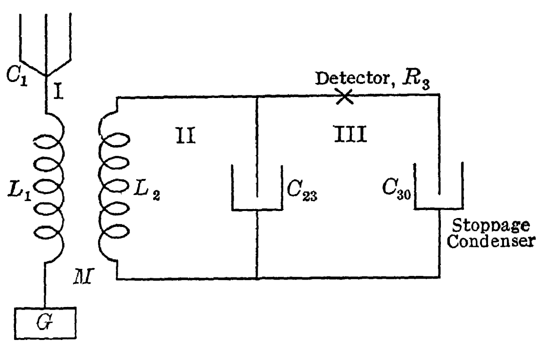

FIG. 1 —Radiotelegraphic receiving circuits with detector in shunt.

the distributed constants of the antenna by a lumped capacity, inductance, and resistance.

It is proposed now to undertake a similar analysis of the corresponding problem with the detector and a "stoppage condenser" C_{30} in shunt to the condenser C_{23} of the secondary circuit, and to attempt to determine under what conditions, if any, this arrangement is superior to the arrangement of Chapter XII, Fig. 1.

The form of circuit constituting the subject matter of the present chapter is given in Fig. 1. If we idealize this circuit by replacing the antenna and ground by a lumped capacity, as was done in the previous chapters, we have the arrangement given in Fig. 2, in which the condenser C_1 replaces the antenna and ground, and the local e.m f. e replaces the e.m.f. impressed by the incident waves. If the waves are persistent and undamped, the current will arrive at a steady state even for the shortest dot made at the sending station. We shall seek, therefore, only the steady-state solution.

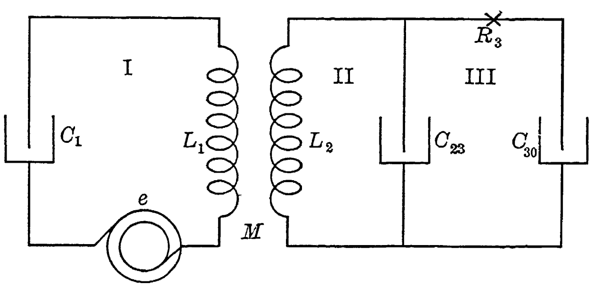

FIG. 2.—Similar to Fig 1, but with antenna circuit replaced by a closed circuit.

221. Notation.—We shall give the various parts of the circuits the designations indicated in Fig. 2. If we compare this notation with that of Fig. 2 of Chapter XIV, we shall see that the notation is the same except that M_{12} has now been simplified to M.

The reactances of the three circuits are seen to be

$$X_1 = L_1\omega - 1/C_1\omega,\ X_2 = L_2\omega - 1/C_{23}\omega,\ X_3 = -1/C_3\omega \quad (1)$$

where

$$\frac{1}{C_3} = \frac{1}{C_{23}} + \frac{1}{C_{30}} \quad (2)$$

Using the methods of the preceding chapters if we let m_{12} and m_{23} be the complex mutual impedances between Circuits I and II and Circuits II and III respectively, and refer to the definition of these quantities given in (18) of Chapter XIII, we see that

$$m_{12} = jM\omega, \quad \text{and} \quad m_{23} = 1/jC_{23}\omega \quad (3)$$

In order now to make Chapters XIII and XIV directly ap-

plicable to the present problem, we shall note that β and γ as used in (9) of Chapter XIV have now the values

$$\beta = M\omega, \quad \gamma = -\frac{1}{C_{23}\omega} \tag{4}$$

With equations (4) as definitions of β and γ, Table I of Chapter XIV (Art. **211**) contains the *Equivalences* for the present case.

222. Current Amplitude I_3 in Circuit III.—By equations (1), (2), and (3) of Chapter XIV, we may now write the current amplitude I_3 in Circuit III in three variant forms as follows:

$$I_3 = \underset{(5)}{\frac{|\beta\gamma E|}{Z_3 Z'_2 Z'_1}} = \underset{(6)}{\frac{|\beta\gamma E|}{Z_1 Z_2^\circ Z_3^\circ}} = \underset{(7)}{\frac{|\beta\gamma E|}{Z_1 Z'^\circ_2 Z_3}} \tag{5,6,7}$$

Equations (5), (6), *and* (7) *give three variant forms of expression for the current amplitude I_3 in Circuit III. In these equations β and γ have the values given in* (4), *and the various Z's have the values given in Table I* Art. **211.**

223. Investigation to Determine the Resonant Values of the Stoppage Condenser C_{30}.—The condenser C_{30} is in practice ordinarily called the *Stoppage Condenser.* We shall now seek the value of C_{30} (called *optimum value*) that gives the greatest current amplitude I_3 in the detector circuit (Circuit III). The detector has any resistance R_3.

If we examine equation (6), we see that β, γ, E, Z_1 and Z_2° are independent of C_{30}, which is involved in Z_3° alone.

The optimum value of C_{30} is thus the value that makes (since Z_3° is positive)

$$Z_3^{\circ 2} = \text{a minimum, re } C_{30} \tag{8}$$

By Table I, Art. **211,**

$$Z_3^{\circ 2} = R_3^{\circ 2} + X_3^{\circ 2} \tag{9}$$

in which, by reference to Table I it is seen that R_3° is independent of C_{30}. We may, therefore, attain our optimum value of C_{30} by making

$$X_3^{\circ 2} = \text{a minimum, re } C_{30} \tag{10}$$

If possible, we shall choose C_{30} to make

$$X_3^\circ = 0 \tag{11}$$

Equation (11), *if attainable, will give the Partial Resonance Relation re C_{30}.*

If, on account of restrictions, it is not possible to fulfill (11), we shall choose C_{30} to make the value of $X_3^{\circ 2}$ a minimum, and obtain what we have called the *Resonance Relation re C_{30} Restricted.*

We shall use the restricted resonance relation only when the partial resonance relation (11) cannot be attained, for if (11) can be attained it will give a larger I_3 than could be had with the restricted relation that does not make X_3° zero.

224. Resonance Relations re C_{30}. Restrictions.—Let us now write down the abbreviated value of X_3° from Table I, Art. **211.** It is

$$X_3^\circ = X_3 - \frac{\gamma^2 X_2^\circ}{Z_2^{\circ 2}} \tag{12}$$

Replacing X_3 by its value from (1) and (2), and indicating the square, we have

$$X_3^{\circ 2} = \left[- \frac{1}{C_{30}\omega} - \frac{1}{C_{23}\omega} - \frac{\gamma^2 X_2^\circ}{Z_2^{\circ 2}} \right]^2 \tag{13}$$

Now C_{30} can have any positive value, so that the first term in the bracket can have any negative value.

We see then that we can make

$$X_3^{\circ 2} = 0 \tag{14}$$

provided the remaining terms in the bracket of (13) are positive; that is provided

$$-\frac{1}{C_{23}\omega} - \frac{\gamma^2 X_2^\circ}{Z_2^{\circ 2}} \geq 0 \tag{15}$$

If (15) *is satisfied, there is some value of C_{30} that satisfies* (14), *and hence* (14) *is attainable and is the resonance relation re C_{30}. In* (15) *X_2° and Z_2° are defined in Table I, Art.* **211.**

If, now, on the other hand, (15) is not satisfied, then the last two terms in the bracket of (13) are negative. The first term in the bracket is also negative, and by inspection it is seen that we shall make the whole bracket squared a minimum, by making the first term zero. Therefore,

for $X_3^{\circ 2}$ a minimum we must make

$$C_{30} = \text{infinity} \tag{16}$$

provided

$$-\frac{1}{C_{23}\omega} - \frac{\gamma^2 X_2^\circ}{Z_2^{\circ 2}} \leq 0 \tag{17}$$

If (17) *is satisfied, equation* (16) *gives the optimum value of C_{30}. This is the Resonance Relation re C_{30} Restricted.*

225. Expansion of Resonance Relations re C_{30}.—We shall now elaborate (14) and (15). To do this, we shall introduce two new abbreviations as follows:

Let

$$A^2 = R_2^2 + L_2^2\omega^2 + \frac{\beta^4}{Z_1^2} + \frac{2\beta^2}{Z_1^2}\left(R_1R_2 - X_1L_2\omega\right) \tag{18}$$

$$B = L_2\omega - \frac{\beta^2 X_1}{Z_1^2} \tag{19}$$

To justify the designation of (18) in a form that is essentially positive, let us note that, if we recall that

$$\frac{\beta^4}{Z_1^2} = \frac{\beta^4(X_1^2 + R_1^2)}{Z_1^4} \tag{20}$$

we can factor (18) into

$$\begin{aligned} A^2 &= \left(R_2 + \frac{\beta^2 R_1}{Z_1^2}\right)^2 + \left(L_2\omega - \frac{\beta^2 X_1}{Z_1^2}\right)^2 \\ &= R_2^{\circ 2} + B^2 \end{aligned} \tag{21}$$

which shows it to be essentially positive.

Now making use of Table I, Art. **211,** and equations (1), (2), (4) and (19), we have

$$\begin{aligned} X_2^\circ &= X_2 - \frac{\beta^2 X_1}{Z_1^2} \\ &= L_2\omega + \gamma - \frac{\beta^2 X_1}{Z_1^2} \\ &= B + \gamma \end{aligned} \tag{22}$$

Also,

$$\begin{aligned} Z_2^{02} &= R_2^{\circ 2} + X_2^{\circ 2} \\ &= R_2^{\circ 2} + B^2 + 2B\gamma + \gamma^2 \\ &= A^2 + 2B\gamma + \gamma^2 \end{aligned} \tag{23}$$

In terms of these results, we can express (14), by using (13), as follows

$$0 = X_3^\circ = -\frac{1}{C_{30}\omega} + \gamma - \frac{\gamma^2(B+\gamma)}{A^2 + 2B\gamma + \gamma^2}$$

Therefore,

$$0 = -\frac{1}{C_{30}\omega} + \frac{\gamma(A^2 + B\gamma)}{A^2 + 2B\gamma + \gamma^2} \tag{24}$$

This gives

$$C_{30}\omega = \frac{1}{\gamma} + \frac{B+\gamma}{A^2 + B\gamma} \tag{25}$$

Replacing γ by its value $-1/C_{23}\omega$, we obtain from (25)

$$C_{30}\omega = -C_{23}\omega + \frac{1 - C_{23}\omega B}{B - C_{23}\omega A^2} \tag{26}$$

Since in (24) the denominator of the last fraction is positive, and since γ is negative, and $C_{30}\omega$ is positive, equation (24) and consequently (26) can be realized, only provided

$$A^2 + B\gamma \gtreqless 0 \tag{27}$$

Replacing γ by its value this last inequality can be replaced by

$$A^2 \gtreqless \frac{B}{C_{23}\omega} \tag{28}$$

Equation (26) *gives the value of* $C_{30}\omega$ *for a maximum amplitude of* I_3. *This is the Partial Resonance Relation re* C_{30}. *It can be attained only provided* (28) *is satisfied.*

If (28) *is not satisfied, we must use the Restricted Resonance Relation, given in* (16); *namely,*

$$C_{30} = \infty \tag{29}$$

We shall consider next the Resonance Relations re C_{23}.

226. Resonance Relations re C_{23}.—We shall now make an independent investigation of the resonance relations re C_{23}, and shall begin with the current amplitude equation (6), which squared gives

$$I_3^2 = \frac{\beta^2\gamma^2E^2}{Z_1^2Z_2^{\circ 2}Z_3^{\circ 2}}, \text{ where } \gamma = -\frac{1}{C_{23}\omega}.$$

In this equation the quantities γ, Z_2° and Z_3° all contain C_{23}, while the other quantities of the equation do not, so that for a maximum I_3^2 with respect to C_{23}, we must make

$$\frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2} = \text{a minimum, re } C_{23} \tag{30}$$

Now, by Table I, Art. **211**,

$$\begin{aligned} Z_3^{\circ 2} &= \left(R_3 + \frac{\gamma^2R_2^\circ}{Z_2^{\circ 2}}\right)^2 + \left(X_3 - \frac{\gamma^2X_2^\circ}{Z_2^{\circ 2}}\right)^2 \\ &= R_3^2 + X_3^2 + \frac{2\gamma^2(R_2^\circ R_3 - X_2^\circ X_3)}{Z_2^{\circ 2}} + \frac{\gamma^4}{Z_2^{\circ 2}}, \end{aligned}$$

whence

$$\begin{aligned} \frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2} &= \frac{Z_2^{\circ 2}}{\gamma^2}\left(R_3^2 + X_3^2\right) + 2\left(R_2^\circ R_3 - X_2^\circ X_3\right) + \gamma^2 \\ &= \frac{(A^2 + 2B\gamma + \gamma^2)(R_3^2 + X_3^2)}{\gamma^2} + \gamma^2 + 2R_2^\circ R_3 \\ &\qquad - 2X_3(B + \gamma). \end{aligned}$$

In this expression X_3 still involves γ, and must be replaced by its value from (1). This gives, after simplification,

$$\frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2} = \frac{1}{\gamma^2}\left\{ A^2\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right)\right\} + \frac{1}{\gamma}\left\{ 2B\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right) - \frac{2A^2}{C_{30}\omega}\right\} + \left(R_2^\circ + R_3\right)^2 + \left\{ B - \frac{1}{C_{30}\omega}\right\}^2 \tag{31}$$

To make this a minimum with respect to γ, let us set the derivative of it with respect to γ equal to zero, obtaining

$$0 = -\frac{1}{\gamma^3}\left\{A^2\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right)\right\} - \frac{1}{\gamma^2}\left\{B\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right) - \frac{A^2}{C_{30}\omega}\right\} \tag{32}$$

whence,

either $$\gamma = -\infty \tag{33}$$

or $$-\frac{1}{\gamma} = \frac{B}{A^2} - \frac{1}{C_{30}\omega\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right)} \tag{34}$$

Since γ is negative, (34) can be attained, only provided

$$\frac{B}{A^2} \gtreqless \frac{1}{C_{30}\omega\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right)} \tag{35}$$

To ascertain whether (33) or (34) gives the larger value of current amplitude I_3, let us substitute these two values successively into (31), and designate the results respectively by D_1 and D_2, as temporary abbreviations, obtaining

$$\frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2} = (R_2^\circ + R_3)^2 + \left(B - \frac{1}{C_{30}\omega}\right)^2 = D_1 \tag{36}$$

when γ has the value given by (33); and (using (32) for (34))

$$\frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2} = -\frac{1}{\gamma^2}A^2\left(R_3^2 + \frac{1}{C_{30}^2\omega^2}\right) + (R_2^\circ + R_3)^2 + \left(B - \frac{1}{C_{30}\omega}\right)^2 = D_2 \tag{37}$$

when γ has the value given by (34). It is seen by inspection that D_2 is less than or equal to D_1, so that (34) gives more current amplitude I_3 than does (33), and is to be used whenever it can be attained; that is, whenever (35) is satisfied.

We may now replace γ in (33) and (34) by its value (4), obtaining

either
$$C_{23} = 0 \tag{38}$$

or
$$C_{23}\omega = \frac{B}{A^2} - \frac{1}{C_{30}\omega R_3^2 + \dfrac{1}{C_{30}\omega}} \tag{39}$$

When (35) *is satisfied, equation* (39) *gives the value of* C_{23} *that produces a maximum value of* I_3. *When* (35) *is not satisfied, equation* (38) *is to be used to obtain a maximum value of* I_3.

227. Optimum Simultaneous Adjustments of Both C_{30} and C_{23}. Resonance Combination L.—We have now obtained independently the optimum adjustments re C_{30} as given in (26) and (29) distinguished by the criterion (28), and the optimum adjustment of C_{23} as given in (38) and (39) distinguished by the criterion (35).

We shall next determine what simultaneous adjustments of both C_{30} and C_{23} are optimum, leaving C_1 still arbitrary.

This is done by treating these various equations as simultaneous, keeping in view the criteria under which any of the respective combinations is attainable.

Let us begin with the combination

and
$$\left.\begin{aligned} C_{30} &= \infty \\ C_{23} &= 0 \end{aligned}\right\} \tag{40}$$

By reference to the descriptive matter concerning (29) and (33) we see that the equations (40) can be a proper resonance combination only provided this combination is inconsistent with (28) and (35). To be inconsistent with these inequalities (28) and (35) we require that

$$A^2 \gtreqless \frac{B}{C_{23}\omega}, \qquad \text{when } C_{23} = 0,$$

and

$$\frac{B}{A^2} \lesseqgtr \frac{1}{C_{30}\omega R_3^2 + \dfrac{1}{C_{30}\omega}}, \text{ when } C_{30} = \infty$$

These two relations merely require that

$$B < 0 \tag{41}$$

We have then the result that under condition (41) *the optimum combination of values of* C_{30} *and* C_{23} *is that given by* (40). *This*

means that in this case C_{30} is short circuited and C_{23} is open circuited or removed. Since in this case the capacities no longer enter, we shall designate this combination (40), *under condition* (41) *as Resonance Combination L.*

228. Optimum Simultaneous Adjustment of C_{30} and C_{23}. Resonance Combination 0.—Let us examine next the combination of (38) with (26) If these two equations are simultaneous we have

$$\left.\begin{aligned} C_{23} &= 0 \\ C_{30}\omega &= \frac{1}{B} \end{aligned}\right\} \tag{42}$$

The restrictions under which the equations (38) and (26) were resonance relations are that (35) be not satisfied and that (28) be satisfied. That is,

$$\frac{B}{A^2} \gtreqless \frac{1}{C_{30}\omega R_3{}^2 + \frac{1}{C_{30}\omega}}, \text{ when } C_{30}\omega = 1/B,$$

and

$$A^2 \gtreqless \frac{B}{C_{23}\omega}, \qquad \text{when } C_{23} = 0.$$

The second of these inequalities gives

$$B > 0 \tag{43}$$

and the first, on replacing $C_{30}\omega$ by $1/B$, and inverting the inequality, gives

$$\frac{A^2}{B} \lesseqgtr \frac{R_3{}^2 + B^2}{B}.$$

This by (43) and (21) gives

$$R_2{}^0 \lesseqgtr R_3 \tag{44}$$

Under conditions (43) *and* (44), *the combination* (42) *is the optimum resonance combination with respect to both C_{30} and C_{23}. We shall call this Resonance Combination* 0.

229. Resonance Combination A.—Let us next investigate (39) and (26) as a possible combination. This requires extensive elimination.

By partial division of the fractional part of (26) this equation gives

$$\begin{aligned} C_{30}\omega &= -C_{23}\omega + \frac{B}{A^2} + \frac{B^2 - A^2}{A^2(C_{23}\omega A^2 - B)} \\ &= -C_{23}\omega + \frac{B}{A^2} + \frac{B^2 - A^2}{A^4\left(C_{23}\omega - \frac{B}{A^2}\right)} \end{aligned} \tag{45}$$

Equation (45) *is the equivalent of* (26).

Let us now replace the first two terms on the right and the corresponding expression in the last denominator by its equivalent from (39), obtaining

$$C_{30}\omega = \frac{1}{C_{30}\omega R_3^2 + \frac{1}{C_{30}\omega}} + \frac{A^2 - B^2}{A^4}\left(C_{30}\omega R_3^2 + \frac{1}{C_{30}\omega}\right).$$

Transposing the first term of the right-hand side to the left, and collecting these two terms over a common denominator, we obtain, after clearing of fractions

$$C_{30}^2\omega^2 R_3^2 = \frac{A^2 - B^2}{A^4}\left(C_{30}\omega R_3^2 + \frac{1}{C_{30}\omega}\right)^2.$$

By (21)

$$A^2 - B^2 = R_2^{\circ 2},$$

which, introduced into the preceding equation, gives a perfect square on both sides. Taking the square root, we obtain

$$C_{30}\omega R_3 = \frac{R_2^\circ C_{30}\omega R_3^2}{A^2} + \frac{R_2^\circ}{A^2 C_{30}\omega}.$$

Clearing this of fractions and solving, we obtain

$$C_{30}\omega = \sqrt{\frac{R_2^\circ}{R_3(A^2 - R_2^\circ R_3)}} \tag{46}$$

The substitution of (46) into (39) gives

$$C_{23}\omega = \frac{B - \sqrt{\frac{R_2^\circ}{R_3}(A^2 - R_2^\circ R_3)}}{A^2} \tag{47}$$

The conditions under which these results can be attained are the conditions that make the radicals real, and make the numerator of (47) positive. These are

$$\left.\begin{array}{l} A^2 \gtreqless R_2^\circ R_3, \text{ and } B > 0 \\ B^2 \gtreqless \dfrac{R_2^\circ A^2}{R_3} - R_2^{\circ 2} \end{array}\right\} \tag{48}$$

By (21) the latter gives

$$R_3 \gtreqless R_2^\circ$$

This equation combined with (48) gives for the complete condition

$$B > 0, \text{ and } R_2^\circ \leqq R_3 \leqq \frac{A^2}{R_2^\circ} \tag{49}$$

Under conditions (49) equations (46) and (47) give the optimum resonance combination with respect to both C_{30} and C_{23}. We shall call this Resonance Combination A.

230. Resonance Combination B.—There remains one other possible combination; namely, the combination of (28) and (39). This combination gives

$$\left.\begin{aligned} C_{30} &= \infty \\ \text{and} \qquad C_{23}\omega &= B/A^2 \end{aligned}\right\} \tag{50}$$

We shall call (50) the Resonance Combination B

Examination shows that the only restriction on this is $B > 0$, so that Resonance Combination B as given in (50) is applicable coextensive with Resonance Combinations 0 and A. It can be shown, however, that where either 0 or A is attainable the Resonance Combination B is inferior as a resonance relation, as follows:

Taking the general equation (31), introducing in turn Combination 0 and Combination B as given in equations (50), and calling the results D_0 and D_B, we have

$$D_0 = (R_2^\circ + R_3)^2 \tag{51}$$

and

$$D_B = \frac{-B^2R_3^2 + A^2B^2}{A^2} + (R_2^\circ + R_3)^2 \tag{52}$$

Let us note for future use that by (21) this can be written

$$D_B = \left(\frac{A^2 + R_2^\circ R_3}{A}\right)^2 \tag{53}$$

Referring now to (51) and (52) it is seen that D_0 is the smaller whenever the fraction of (52) is positive; that is, whenever

$$R_3^2 < A^2.$$

Since by (21), the right-hand side is greater than $R_2^{\circ 2}$, we have *a fortiori* that D_0 is smaller, when

$$R_3 \lesseqgtr R_2^\circ \tag{54}$$

It thus appears that 0 gives a larger I_3 than does (B), whenever 0 is applicable, as may be seen by comparing (54) with (44).

We shall next show that Resonance Combination B as given in (50) is inferior to Resonance Combination A, whenever A is attainable. This is done by comparing the current I_3 at Com-

bination A with that at the combination given in (50). Combination A is given in (46) and (47). By (46)

$$\frac{1}{C_{30}{}^2\omega^2} = \frac{R_3A^2}{R_2{}^\circ} - R_3{}^2 \tag{55}$$

whence by transposition

$$R_3{}^2 + \frac{1}{C_{30}{}^2\omega^2} = \frac{R_3A^2}{R_2{}^\circ} \tag{56}$$

Equation (56) *is an alternative statement of* (46).

Let us next note by transposition of (47) and multiplication by (46), that we obtain

$$(A^2C_{23}\omega - B)C_{30}\omega = -R_2{}^\circ/R_3,$$

whence

$$-C_{23}\omega = \frac{1}{A^2}\left(\frac{R_2{}^\circ}{R_3}\frac{1}{C_{30}\omega} - B\right) = \frac{1}{\gamma} \tag{57}$$

Equation (57) *is a partial expression of Combination B, and is true whenever* (46) *and* (47) *are true.*

Introducing (56) and (57) into (37), we obtain on expanding terms

$$D_2 = -\frac{R_2{}^\circ}{R_3C_{30}{}^2\omega^2} + \frac{2B}{C_{30}\omega} - \frac{R_3B^2}{R_2{}^\circ} + (R_2{}^\circ + R_3)^2 + B^2 - \frac{2B}{C_{30}\omega} + \frac{1}{C_{30}{}^2\omega^2},$$

which reduces to

$$D_2 = \frac{1}{C_{30}{}^2\omega^2}\left\{1 - \frac{R_2{}^\circ}{R_3}\right\} + (R_2{}^\circ + R_3)^2 + B^2 - \frac{R_3B^2}{R_2{}^\circ}$$

In this, let us replace the first factor on the right by its value from (55), obtaining

$$D_2 = \frac{R_3A^2}{R_2{}^\circ} - A^2 - R_2{}^2 + R_2{}^\circ R_3 + (R_2{}^\circ + R_3)^2 + B^2 - \frac{R_3B^2}{R_2{}^\circ}.$$

Now making use of (21) this may be reduced to

$$D_2 = 4R_2{}^\circ R_3.$$

Identifying D_2 as the first member of (37), we have

$$\frac{Z_2{}^{\circ 2}Z_3{}^{\circ 2}}{\gamma^2} = 4R_2{}^\circ R_3 = D_2 \tag{58}$$

Equation (58) *is the value assumed by the general equation* (37) *whenever the Resonance Relations A are fulfilled.*

We shall now show that the right-hand side of (58), which is obtained with Resonance Combination A is smaller than the corresponding expression obtained with the Resonance Combination B given in (50). We have already found that the result obtained with Combination B is

$$\frac{Z_2^{\circ 2} Z_3^{\circ 2}}{\gamma^2} = D_B \tag{59}$$

where D_B has the value given in (53). We see then that

$$D_2 \leqq D_B,$$

whenever

$$\left(\frac{A^2 + R_2^\circ R_3}{A}\right)^2 - 4R_2^\circ R^3 > 0$$

This inequality reduces to

$$(A^2 - R_3^\circ R_3)^2 > 0,$$

which is always fulfilled.

We have then the result that the denominator (proportional to D_2) in the expression for I_3 is less with the Combination A than that (proportional to D_B) with the Combination B, so that whenever Combination A can be realized it is to be preferred to Combination B.

We have then the result that Resonance Combination B is to be used only when B is greater than zero, and when neither Combination 0 nor Combination A can be fulfilled.

An examination of this fact leads to the conclusion that Resonance Combination B as given in equations (50) is valid only when

$$B > 0, \text{ and } \frac{A^2}{R_2^\circ} \gtreqless R_3 \tag{60}$$

Before summing up these results in a Key, let us obtain expressions for the current amplitude I_3 for these several Resonance Combinations, L, 0, A, and B.

231. Amplitude of Current I_3 for Resonance Combinations L, 0, A, and B.—To obtain expressions for the amplitude of current for these several resonance combinations, we may employ equation (6), which squared may be written

$$I_3^2 = \frac{M^2\omega^2E^2}{Z_1^2\left(\frac{Z_2^{\circ 2}Z_3^{\circ 2}}{\gamma^2}\right)} = \frac{M^2\omega^2E^2}{Z_1^2D} \tag{61}$$

where, as a temporary abbreviation,

$$D = \frac{Z_2^{\circ 2} Z_3^{\circ 2}}{\gamma^2} \tag{63}$$

In case of the Resonance Combination L, we can find D, which we shall then call D_L, by substituting (40) into (31), bearing in mind that

$$1/\gamma = -C_{23}\omega \tag{64}$$

This gives

$$D_L = (R_2^\circ + R_3)^2 + B^2,$$

so that by (61) the current in this case becomes

$$[I_{3\,\text{max. max.}}]_L = \frac{M\omega E}{Z_1\sqrt{(R_2^\circ + R_3)^2 + B^2}} \tag{64a}$$

Equation (64a) gives the Current Amplitude in Circuit III for the Resonance Combination L.

To get the current for Resonance Combination 0, we have already obtained D in the form of D_0 in equation (51), so that by (61)

$$[I_{3\,\text{max max.}}]_0 = \frac{M\omega E}{Z_1(R_2^\circ + R_3)} \tag{65}$$

This is the current amplitude for Resonance Combination 0.

Likewise for Resonance Combination A, we use the value of D given in equation (58) and obtain

$$[I_{3\,\text{max max}}]_A = \frac{M\omega E}{2Z_1\sqrt{R_2^\circ R_3}} \tag{66}$$

This is the current amplitude for Resonance Combination A.

For Resonance Combination B, we use the value of D given as D_B in (53), and obtain

$$[I_{3\,\text{max max.}}]_B = \frac{M\omega E}{Z_1\left(A + \frac{R_2^\circ R_3}{A}\right)} \tag{67}$$

This is the current amplitude for Resonance Combination B.

232. Summary and Key to Results for Optimum Values of C_{23} and C_{30} and for Maximum Values of I_3, with Arbitrary Values of X_1.—We are now prepared to give a summary of results obtained up to the present. In this we shall use the following abbreviations, which have already been defined:

$$A^2 = R_2^2 + L_2^2\omega^2 + \frac{M^4\omega^4}{Z_1^2} + \frac{2M^2\omega^2}{Z_1^2}(R_1R_2 - X_1L_2\omega) \quad (68)$$
(see (18)).

$$B = L_2\omega - \frac{M^2\omega^2X_1}{Z_1^2} \quad \text{(see (19))}, \quad (69)$$

$$R_2^\circ = R_2 + \frac{M^2\omega^2R_1}{Z_1^2} \quad \text{(see Table I, Art. 211)} \quad (70)$$

Among these quantities there exists the relation

$$A^2 = R_2^{\circ 2} + B^2 \quad \text{(see (21))} \quad (70a)$$

A key of optimum relations and amplitudes now follows.

I. If $B \leqq 0$, Resonance Combination L,

L. The optimum C_{23} and C_{30} are

$$C_{23} = 0, C_{30} = \infty \quad (71)$$

and the max. max. current is

$$[I_{3\,\max\,\max}]_L = \frac{M\omega E}{Z_1\sqrt{(R_2^\circ + R_3)^2 + B^2}} \quad (72)$$

II. If $B \gtreqqless 0$, there are three combinations, 0, A, and B.

0. When $R_3 \gtreqqless R_2^\circ$,

the optimum C_{23} and C_{30} are

$$C_{23} = 0, \quad C_{30}\omega = 1/B \quad (73)$$

and the max. max. current is

$$[I_{3\,\max\,\max.}]_0 = \frac{M\omega E}{Z_1(R_2^\circ + R_3)} \quad (74)$$

A. When $R_2^\circ \leqq R_3 \leqq \dfrac{A^2}{R_2^\circ}$,

the optimum C_{23} and C_{30} are

$$\left.\begin{aligned} C_{23}\omega &= \frac{B - \sqrt{\dfrac{R_2^\circ}{R_3}(A^2 - R_2^\circ R_3)}}{A^2} \\ C_{30}\omega &= \sqrt{\frac{R_2^\circ}{R_3(A^2 - R_2^\circ R_3)}} \end{aligned}\right\} \quad (75)$$

and the max. max. current is

$$[I_{3\,\max.\,\max}]_A = \frac{M\omega E}{2Z_1\sqrt{R_2^\circ R_3}} \quad (76)$$

B. When $\dfrac{A^2}{R_2{}^\circ} \gtreqless R_3$,

the optimum C_{23} and C_{30} are

$$C_{23}\omega = B/A^2, \quad C_{30} = \infty \tag{77}$$

and the max. max. current is

$$[I_{3\,\max\,\max}]_B = \frac{M\omega E}{Z_1\left(A + \dfrac{R_2{}^\circ R_3}{A}\right)} \tag{78}$$

These equations and the several criteria under which the equations are applicable are given in terms of quantities A, B, $R_2{}^\circ$, and Z_1, all of which involve X_1. For any given X_1 the criteria in the form of inequalities enable us to select the proper Resonance Combination and to compute the value of $I_{3\,\max\,\max}$.

233. Abbreviations in the Form of Ratio Quantities.—For purposes of calculation, it is desirable to introduce into the previous equations certain ratios of the obvious electrical constants or variables of the circuits. As in previous chapters let

$$\tau^2 = \frac{M^2}{L_1L_2}, \quad \eta_1 = \frac{R_1}{L_1\omega}, \quad \eta_2 = \frac{R_2}{L_2\omega} \tag{79}$$

$$\Omega_1{}^2 = \frac{1}{L_1C_1}, \quad \Omega_2{}^2 = \frac{1}{L_2C_{23}}, \quad \Omega_3{}^2 = \frac{1}{L_2C_{30}} \tag{80}$$

The last of these is a new ratio, taking account of the third condenser of the system, and combining it arbitrarily with the second inductance.

In addition to these ratios let us employ also the following:

$$J_1 = 1 - \frac{\Omega_1{}^2}{\omega^2} = \frac{X_1}{L_1\omega} \tag{81}$$

$$b = \frac{B}{L_2\omega} = 1 - \frac{\tau^2 J_1}{r^2} \tag{82}$$

where

$$r^2 = \frac{Z_1{}^2}{L_1{}^2\omega^2} = \eta_1{}^2 + J_1{}^2 \tag{83}$$

$$a^2 = \frac{A^2}{L_2{}^2\omega^2} = 1 + \eta_2{}^2 + \frac{\tau^4}{r^2} + \frac{2\tau^2(\eta_1\eta_2 - J_1)}{r^2} \tag{84}$$

$$\rho = \frac{R_3}{R_2} \tag{85}$$

In some of the computations we shall replace also the inverse ratios of angular velocities by ratios of wavelengths, by writing

$$\frac{\Lambda_1}{\lambda} = \frac{\omega}{\Omega_1}, \quad \frac{\Lambda_2}{\lambda} = \frac{\omega}{\Omega_2}, \quad \frac{\Lambda_3}{\lambda} = \frac{\omega}{\Omega_3} \tag{86}$$

where

λ = wavelength of impressed e.m.f.,
$\Lambda_1, \Lambda_2, \Lambda_3$ = undamped wavelengths corresponding to the undamped angular velocities $\Omega_1, \Omega_2, \Omega_3$, respectively.

234. Summary and Key in Terms of Ratio Quantities.—In terms of this set of ratio quantities, the summary given two sections back can now be put into the following forms suitable for computations:

I. If $b \leqq 0$,

L. Use the resonance relations

$$C_{23} = 0, \; C_{30} = \infty \tag{87}$$

and calculate the current by

$$[I_{3_{\max\max}}]_L = \frac{\tau E}{\sqrt{R_1 R_3}\sqrt{\dfrac{\left\{\eta_2 r\left(\dfrac{1}{\sqrt{\rho}} + \sqrt{\rho}\right) + \dfrac{\tau^2\eta_1}{r\sqrt{\rho}}\right\}^2 + \dfrac{r^2 b^2}{\rho}}{\eta_1\eta_2}}} \tag{88}$$

II. If $b \geqq 0$,

0. When

$$\rho \leqq 1 + \frac{\tau^2\eta_1}{r^2\eta_2},$$

use the resonance relations

$$C_{23} = 0, \text{ and } \frac{\omega^2}{\Omega_3{}^2} = \frac{\Lambda_3{}^2}{\lambda^2} = \frac{1}{b} \tag{89}$$

and calculate the current by

$$[I_{3_{\max\max}}]_0 = \frac{\tau E}{\sqrt{R_1 R_3}\left\{r\left(\dfrac{1}{\sqrt{\rho}} + \dfrac{\sqrt{\rho}}{1}\right)\sqrt{\dfrac{\eta_2}{\eta_1}} + \dfrac{\tau^2}{r\sqrt{\rho}}\sqrt{\dfrac{\eta_1}{\eta_2}}\right\}} \tag{90}$$

A. When

$$1 + \frac{\tau^2\eta_1}{r^2\eta_2} \leqq \rho \leqq \frac{a^2}{\eta_2{}^2\left(1 + \dfrac{\tau^2\eta_1}{r^2\eta_2}\right)},$$

use the resonance relations

$$\frac{\omega^2}{\Omega_2{}^2} = \frac{\Lambda_2{}^2}{\lambda^2} = \frac{b - \sqrt{\frac{1}{\rho}\left\{1 + \frac{\tau^2\eta_1}{r^2\eta_2}\right\}\left\{a^2 - \rho\eta_2\left(\eta_2 + \frac{\tau^2\eta_1}{r^2}\right)\right\}}}{a^2} \tag{91}$$

$$\frac{\omega^2}{\Omega_3{}^2} = \frac{\Lambda_3{}^2}{\lambda^2} = \sqrt{\frac{\left(1 + \frac{\tau^2\eta_1}{r^2\eta_2}\right)}{\rho\left\{a^2 - \rho\eta_2\left(\eta_2 + \frac{\tau^2\eta_1}{r^2}\right)\right\}}} \tag{92}$$

and calculate the current by

$$[I_{3_{\text{max. max.}}}]_A = \frac{\tau E}{2\sqrt{R_1R_3}\sqrt{\frac{\eta_2}{\eta_1}r^2 + \tau^2}} \tag{93}$$

B. When

$$\frac{a^2}{\eta_2{}^2\left(1 + \frac{\tau^2\eta_1}{r^2\eta_2}\right)} \leqq \rho,$$

use the resonance relations

$$\frac{\omega^2}{\Omega_2{}^2} = \frac{\Lambda_2{}^2}{\lambda^2} = \frac{b}{a^2},\ C_{30} = \infty \tag{94}$$

and calculate the current by

$$[I_{3_{\text{max. max}}}]_B = \frac{\tau E}{\sqrt{R_1R_3}\,\frac{r}{\sqrt{\rho\eta_1\eta_2}}\left[a + \frac{\rho\left(\eta_2{}^2 + \tau^2\frac{\eta_1\eta_2}{r^2}\right)}{a}\right]} \tag{95}$$

In terms of the abbreviations (81) *to* (87) *the several equations of this summary give the relations for calculating the optimum adjustments of C_{23} and C_{30}, for any given value of X_1, or the related quantity J_1. There are contained also in the summary the values of $I_{3_{\text{max. max.}}}$ obtained when these respective adjustments are made.*

Before proceeding to a theoretical determination of the adjustment also of the Circuit I to give what may be called a grand maximum of current, we shall give an illustration of the results up to the present by the aid of numerical computations.

II. COMPUTATIONS IN A SPECIAL CASE

235. Power Developed in the Detector for Various Adjustments of the Primary Circuit, with Optimum Adjustment of Secondary Condenser C_{23} and Stoppage Condensers C_{30}, in a Special Case in Which $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$.—If we take the squares of the current equations (88), (90), (93) and (95)

and multiply them by R_1R_3, we may obtain values of $I_3{}^2R_1R_3/\tau^2E^2$, which are proportional to the power developed in the detector, whose resistance is R_3. This we shall do in a series of special cases in all of which

$$\left.\begin{aligned} \tau^2 &= 0.1 \\ \eta_1 &= 0.03 \\ \eta_2 &= 0.01 \\ \text{with} \qquad \frac{R_3}{R_2} &= 10^4,\ 10^3,\ 10^2 \text{ and } 10 \end{aligned}\right\} \tag{96}$$

The values of τ, η_1, and η_2 are approximately those attainable in practice in radiotelegraphic receiving. As to the resistance R_3 of the detector, reliable experimental values of this quantity are not at present available, and in fact this resistance is a function of the current, and is complicated by an action of rectification. Nevertheless, it is possible that experiment may subsequently separate out from the complicated action of the detectors a term of the character of pure resistance, and also new types of detectors, more nearly approaching constancy of resistive action, may be discovered. These calculations may then be of great importance in pointing the way to proper design of receiving apparatus.

Table I gives a series of calculation of relative power developed in the detectors of various resistance R_3 relative to R_2. The quantity called relative power is arbitrarily defined as follows:

$$\text{Relative Power} = \frac{I_3{}^2R_3R_1}{\tau^2E^2} \tag{97}$$

Table I was made as follows: Taking various arbitrary adjustments of J_1 of the primary circuit, values of Ω_1/ω were computed by (81). The result was put into terms of relative wavelengths, by employing the relation

$$\frac{\Lambda_1}{\lambda} = \frac{\omega}{\Omega_1} \tag{98}$$

where Λ_1 is the *undamped, or forced wavelength,* defined in Art. **66**, Chapter VI. The values of the generalized wavelength divided by the impressed wavelength λ, corresponding to the assumed values of J_1 are put into the first column of the table (Table I). Next, corresponding to the various values of J_1, the several

Table I.—Power Developed in the Detector R_3 at Optimum Adjustment of C_{23} and C_{30} for Various Settings of the Antenna Undamped Wavelength Λ_1 Relative to the Incident Wavelength λ and for Various Ratios of Detector Resistance to Secondary Resistance. Given $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$

$\frac{\Lambda_1}{\lambda}$	J_1	Relative power developed in R_3			
		$\frac{R_3}{R_2} = 10^4$	10^3	10^2	10
∞	1 0	0 548 *B*	0 576*A*	0 576*A*	0 576*A*
1 58	0 6	0 889 *B*	1 14 *A*	1 14 *A*	1 14 *A*
1 41	0 5	0 941 *B*	1 37 *A*	1 37 *A*	1 37 *A*
1 29	0.4	0 889 *B*	1 62 *A*	1.62 *A*	1 62 *A*
1 20	0 3	0 691 *B*	1 92 *A*	1.92 *A*	1 92 *A*
1 12	0 2	0 265 *B*	1 63 *B*	2.20 *A*	2 20 *A*
1 104	0 18	0 191 *B*	1 35 *B*	2.25 *A*	2 25 *O*
1 084	0.15	0 095 *B*	0 80 *B*	2.33 *A*	2 26 *O*
1 05	0 1	0 0292*B*	0 275*B*	1.74 *B*	1 86 *O*
1 045	0 09	0 0331*L*	0 313*L*	1.85 *L*	1 71 *L*
1 04	0.08	0 0408*L*	0.380*L*	2.05 *L*	1 473*L*
1 03	0 06	0.0658*L*	0 585*L*	2.30 *L*	0.935*L*
1 02	0.04	0.117 *L*	0 952*L*	2.31 *L*	0 578*L*
1 01	0.02	0 221 *L*	1.52 *L*	2.05 *L*	0 377*L*
1 005	0.01	0.283 *L*	1.78 *L*	1.87 *L*	0 316*L*
1 00	0.00	0 367 *B*	1.95 *B*	1.76 *O*	0.284*O*
0 976	−0.05	0 660 *B*	2 47 *B*	2.47 *A*	0 894*O*
0 967	−0.07	0 774 *B*	2.45 *B*	2 45 *A*	1.30 *O*
0 953	−0.10	1.00 *B*	2 40 *A*	2 40 *A*	1.86 *O*
0.933	−0.15	1.24 *B*	2.33 *A*	2 33 *A*	2 26 *O*
0.921	−0.18	1.40 *B*	2.25 *A*	2 25 *A*	2.25 *O*
0 913	−0 20	1.46 *B*	2.22 *A*	2.22 *A*	2.22 *A*
0.877	−0.3	1.59 *B*	1.92 *A*	1.92 *A*	1.92 *A*
0 850	−0 4	1.49 *B*	1.62 *A*	1.62 *A*	1 62 *A*
0 816	−0 5	1.30 *B*	1.37 *A*	1 37 *A*	1 37 *A*
0 792	−0.6	1.12 *B*	1.14 *A*	1.14 *A*	1 14 *A*
0 707	−1.0	0.573 *B*	0.576*A*	0.576*A*	0.576*A*

criteria of the "Key" in terms of ratio constants were investigated, and the proper formulas for the computation of $I_{3\ \text{max. max.}}$ were selected, thereby imposing upon the system the requirement of an optimum adjustment of C_{23} and C_{30}. By (97) the relative power was then computed, and placed in the last four columns of the table, with each numerical value designated by a letter indicating the formula employed in the calculation.

236. Discussion of Results for Relative Power.—The results are plotted in Fig. 3, with relative power as ordinates and relative primary wavelength as abscissæ. The separate curves marked respectively 10^4, 10^3, 10^2, and 10 are for the ratio of resistances R_3/R_2 equal to these values respectively. It is to be noticed that each of these curves has two maxima, except the 10^2-curve, which has three maxima. Various parts of the

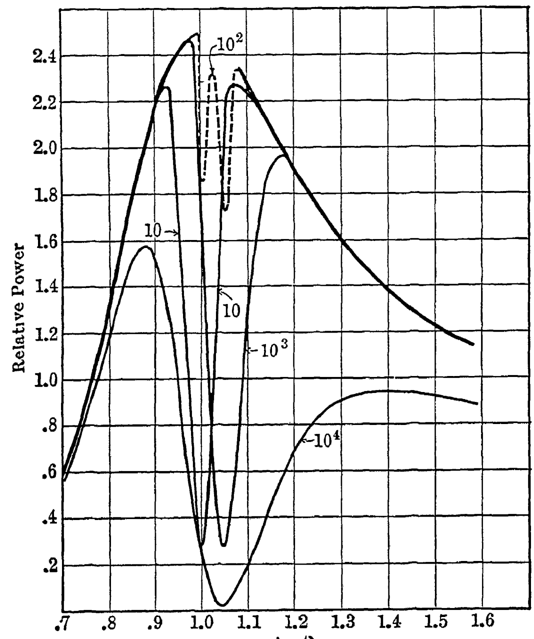

FIG. 3.—Plot of Table I. The quantities 10, 10^2, 10^3 and 10^4 give the values of R_3/R_2 for the separate curves.

various curves of this figure (Fig. 3) were computed by various formulas, in accordance with the criteria relations of the "Key." The heavy black line serving as a sort of upper boundary of the figure was computed by the formula corresponding to Case *A*. In Case *A* the computed value is the same for all ratios R_3/R_2 of resistances, so that wherever the criterion of Case *A* is satisfied by any adjustment of the circuits, the curve obtained comes into coincidence with this heavy bounding line. Each of the curves marked 10, 10^2 and 10^3 has a maximum near its junction with the *A*-curve. The curve marked 10^4 does not have any *A*-values

within the range considered. The curve marked 10^2 has its third maximum (the middle one) on a part of the curve calculated by the L-formula. This is very interesting, for in this region the condensers of the secondary and tertiary circuit are inoperative, one being zero and the other infinite, or short circuited. For

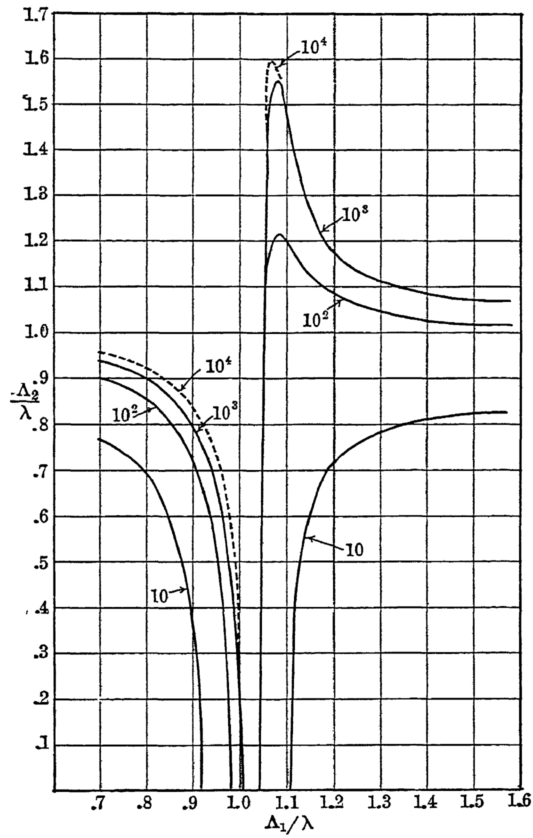

FIG. 4.—Optimum values of Λ_2/λ for various values of Λ_1/λ. The 10, 10^2, 10^3, 10^4 attached to the various curves gives the value of R_3/R_2 for each curve.

this particular set of constants we have an efficient tuning system without any secondary condensers!

Certain other facts regarding these curves will be presented in a theoretical discussion to follow a presentation of tables and graphs of the optimum resonance relations in our special case.

237. The Resonance Relations in the Special Numerical Case. It is proposed now to give numerical results concerning the

optimum adjustments of C_{23} and C_{30} in the special case under consideration. Instead of tabulating the capacities it is more convenient to tabulate Λ_2/λ and Λ_3/λ, where

$$\Lambda_2/\lambda = 2\pi c\sqrt{L_2C_{23}}/\lambda \tag{99}$$

$$\Lambda_3/\lambda = 2\pi c\sqrt{L_2C_{30}}/\lambda \tag{100}$$

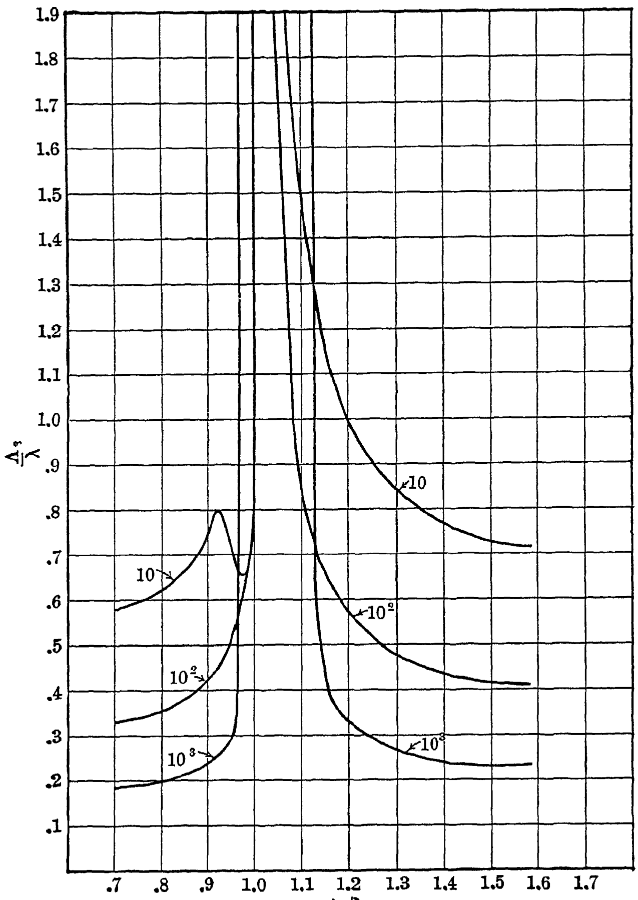

FIG 5.—Optimum values of Λ_3/λ for various values of Λ_1/λ, for different values of R_3/R_2 as designated by numbers attached to the separate curves.

With these definitions it is seen, as has been repeatedly pointed out,

$$\Lambda_2/\lambda = \omega/\Omega_2 \tag{101}$$

$$\Lambda_3/\lambda = \omega/\Omega_3 \tag{102}$$

These values are computed by the aid of the formulas for the resonance relations in the various cases given in the "Key," and are tabulated in Tables II, III, and IV. The results are plotted in Figs. 4 and 5.

The several curves are numbered with numbers giving the ratio of R_3/R_2 taken as the bases for the calculations.

No especial comment will be given, except that these curves permit a determination of the optimum value of the two con-

Table II.—Resonance Relations in Case $R_3/R_2 = 10^4$. Optimum Values of Λ_2/λ and of Λ_3/λ, for Various Values of Λ_1/λ. Given $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$

Given		Calculated optimum values		
Λ_1/λ	J_1	Λ_2/λ	Λ_3/λ	Formula
0 707	−1 00	0 953		
0.808	−0 60	0 927		
0 815	−0 50	0 912		
0 845	−0 40	0 895		
0 877	−0 30	0 866	Infinite	*B*
0 912	−0 20	0 818		
0.953	−0 10	0 714		
0 976	−0 05	0 598		
1 000	0 00	0 285		
1 005	0 01	0 000		
1 01	0 02			
1.03	0.06			
1 036	0 07	Zero	Infinite	*L*
1.042	0 08			
1 046	0 085			
1 049	0 091	0 397		
1 051	0 095	0 657		
1 053	0 100	0 968		
1 061	0 110	1 325		
1 068	0 125	1 60		
1 085	0 150	1 56	Infinite	
1.116	0 200	1 38		*B*
1.194	0.300	1.22		
1 290	0 400	1.153		
1 414	0 500	1.118		
1.570	0 600	1.095		
∞	1.000	1.053		

densers C_{23} and C_{30} in the several numerical cases, and point the way to a further theoretical examination following.

Table III.—Case $R_3/R_2 = 10^3$, 10^2, and 10. Optimum Values of Λ_2/λ For Various Values of Λ_1/λ. Given $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$

Given		Calculated optimum values		
Λ_1/λ	J_1	Values of Λ_2/λ for R_3/R_2 equal		
		10^3	10^2	10
∞	1 0	1 035*A*	0 993*A*	0 842*A*
1.58	0 6	1 072*A*	1 020*A*	0 829*A*
1 41	0 5	1 090*A*	1 030*A*	0 819*A*
1.29	0.4	1 122*A*	1 053*A*	0 783*A*
1 20	0.3	1 170*A*	1 086*A*	0 717*A*
1.12	0 2	1 380*B*	1 166*A*	0 382*A*
1 104	0 18	1 490*B*	1 190*A*	0 000*O*
1 084	0 15	1 556*B*	1 217*A*	0 000*O*
1 05	0 10	0 968*B*	0 968*B*	0 000*O*
1 045	0 09			
1 04	0 08			
1 03	0 06	Zero *L*	Zero *L*	Zero *L*
1 02	0 04			
1 01	0 02			
1 005	0 01			
1 00	0 00	0 287*B*	0 000*O*	0 000*O*
0 976	−0 05	0 565*B*	0 135*A*	0 000*O*
0 967	−0 07	0 632*B*	0 352*A*	0.000*O*
0 953	−0 10	0 650*A*	0 482*A*	0 000*O*
0 933	−0 15	0 732*A*	0 616*A*	0 000*O*
0 921	−0 18	0 762*A*	0 662*A*	0 000*O*
0 913	−0.20	0 781*A*	0 691*A*	0 242*A*
0 877	−0 30	0 838*A*	0.773*A*	0 510*A*
0 850	−0 40	0 870*A*	0 814*A*	0 609*A*
0 816	−0 50	0 892*A*	0 844*A*	0 666*A*
0 792	−0 60	0 907*A*	0.861*A*	0 701*A*
0 707	−1 00	0.936*A*	0 897*A*	0 764*A*

The formula used in each case is that given by the letter following the number given in the table.

Table IV.—Case $R_3/R_2 = 10^3$, 10^2, and 10. Optimum Values of Λ_3/λ for Various Values of Λ_1/λ. Given $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$

Given		Calculated		
		Values of Λ_3/λ for R_3/R_2 equal		
Λ_1/λ	J_1	10^3	10^2	10
∞	1 0	0 200*A*	0 356*A*	0 630*A*
1 58	0 6	0 226*A*	0 403*A*	0 716*A*
1 41	0 5	0.237*A*	0 430*A*	0 765*A*
1 29	0 4	0.267*A*	0 474*A*	0 842*A*
1 20	0 3	0.313*A*	0 556*A*	0 992*A*
1 12	0 2	∞ *B*	0 748*A*	1 33 *A*
1 104	0 18	∞ *B*	0 820*A*	1.50 *O*
1 084	0 15	∞ *B*	0 948*A*	1 61 *O*
1 05	0 10	∞ *B*	∞ *B*	3 48 *O*
1 045	0.09			
1 04	0 08			
1 03	0 06	∞ *L*	∞ *L*	∞ *L*
1 02	0 04			
1 01	0 02			
1 005	0 01			
1 00	0 00	∞ *B*	1 00 *O*	1 00 *O*
0 976	−0 05	∞ *B*	0 620*A*	0 636*O*
0 967	−0 07	∞ *B*	0 578*A*	0 673*O*
0 953	−0 10	0 295*A*	0 525*A*	0 722*O*
0 933	−0 15	0 267*A*	0 476*A*	0 780*O*
0 921	−0 18	0 254*A*	0.456*A*	0 807*O*
0 913	−0 20	0 247*A*	0 440*A*	0 784*A*
0 877	−0 30	0.224*A*	0 395*A*	0 702*A*
0 850	−0 40	0.207*A*	0 369*A*	0 655*A*
0 816	−0 50	0.196*A*	0 354*A*	0 625*A*
0 792	−0 60	0.191*A*	0 341*A*	0 606*A*
0 707	−1 00	0.181*A*	0 324*A*	0 572*A*

The formula used in each case is that given by the letter following the number in the table.

III. THEORETICAL INVESTIGATION OF THE GRAND MAXIMA OF POWER

238. General Note on Grand Maxima of Power in the Detector. An examination of Table I gives some notion of the adjustment for a grand maximum of power in the detector. In the first

place the values in the table presuppose that the optimum adjustments of C_{23} and C_{30} have been made, and the numbers in the last four columns are max. max. values of relative power, so that the maxima of the several values give max. max. max. relative power. To avoid the use of the term max. max. max. we shall call these values the *grand maxima.*

The table shows that when there are any B-values, the grand maxima seem to fall on the B-sections of the curve or at a point near the junction of the B-section with the A-section. When there are no B-values, the grand maximum seems to fall on the 0-section near its junction with the A-section. In one of the cases there is a third grand maximum on the L-section of the curve corresponding to R_3/R_2 equal to 10^2. These inferences from the special-case curves are now to be corroborated by a theoretical investigation, in which the actual values of the grand maxima of power are to be discovered.

239. Investigation of Grand Maximum of Power with Respect to Resonance Combination L.—Let us designate the relative power developed in the detector R_3 by the letter H, defined as in equation (97); that is

$$H = \text{Relative Power} = \frac{I_3^2 R_1 R_3}{\tau^2 E^2} \tag{103}$$

Comparing this definition with (88), it will be seen that, for Resonance Combination L,

$$H_L = \frac{1}{\dfrac{\left\{\eta_2 r\left(\dfrac{1}{\sqrt{\rho}} + \sqrt{\rho}\right) + \dfrac{\tau^2\eta_1}{r\sqrt{\rho}}\right\}^2 + \dfrac{r^2 b^2}{\rho}}{\eta_1\eta_2}} \tag{104}$$

In this expression r and b involve the reactance constants of Circuit I. We propose now to find the value of X_1 (or of the related quantity J_1) that will make H_L a maximum, and we shall then determine the magnitude of this maximum value, which we shall call the grand maximum with respect to Resonance Combination L.

Whatever adjustment makes I_3^2 a maximum, with τ, R_2, R_3 and E fixed, will make H_L a maximum under the same conditions.

Referring to equation (5) and noting that in that equation Z'_1 alone involves X_1, and that in consequence the square of

the current is made a maximum with respect to X_1, by making X'^2_1 a minimum, we have

$$X'^2_1 = \text{a minimum} \tag{105}$$

for the determination of X_1.

Writing out X'_1 by values from Table I, Art. **211,** we have

$$X'_1 = X_1 - \frac{\beta^2(X_2Z_3^2 - \gamma^2X_3)}{Z_2^2Z_3^2+\gamma^4+2\gamma^2(R_2R_3-X_2X_3)} \tag{106}$$

This must be solved as simultaneous with (87), in order to have all three variables of the circuit made simultaneously optimum, in those cases in which (87) is an optimum condition.

We shall first make $C_{30} = \infty$, and $C_{23} = -1/\gamma$, following definition (4), and shall then make γ approach minus infinity. The first step of this operation gives

$$X'_1 = X_1 - \frac{\beta^2(L_2\omega R_3^2 + L_2\omega\gamma^2 + R_3^2\gamma)}{R_2^2R_3^2 + \gamma^2R_2^2 + R_3^2(L_2^2\omega^2 + 2L_2\omega\gamma + \gamma^2) + \gamma^2L_2^2\omega^2 + 2\gamma^2R_2R_3} \tag{107}$$

As a second step, it is to be noted that as γ approaches negative infinity this expression approaches as a limit

$$X'_1 = X_1 - \frac{\beta^2L_2\omega}{(R_2 + R_3)^2 + L_2^2\omega^2} \tag{108}$$

The minimum value of X'^2_1 is then seen to be zero, which may be always attained if the condenser of the Circuit I is capable of taking all possible values. Setting X'_1 equal to zero, replacing β^2 by its value from (4), and dividing by $L_1\omega$, we obtain

$$J_1 = \frac{\tau^2}{\eta_2^2(1 + \rho)^2 + 1} \tag{109}$$

Equation (109) *gives the value of* J_1 *that produces largest Relative Power in the Detector, when the Resonance Relation L is fulfilled by* C_{23} *and* C_{30}.

The magnitude of the grand maximum of relative power, obtained by substituting (109) into (104) is

$$[H_{\max.}]_L = \frac{1}{\dfrac{\eta_1\theta}{\eta_2\rho} + \dfrac{\tau^4}{\rho\eta_1\eta_2} + 2\tau^2\left(\dfrac{1}{\rho} + 1\right) - \dfrac{\tau^4}{\rho\eta_1\eta_2\theta}} \tag{110}$$

where

$$\theta = 1 + \eta_2^2(\rho + 1)^2 \tag{111}$$

In these equations

$$\rho = R_3/R_2 \tag{112}$$

Equation (109) *gives the value of* J_1 *at which occurs maximum power with the Resonance Combination L, and the value of the relative power at this maximum is given by* (110).

240. Investigation of Grand Maximum of Power with Respect to Resonance Combination 0.—For this combination, by (73),

$$C_{23} = 0, \quad C_{30}\omega = 1/B \tag{113}$$

In this case by (1) and (4) we may write

$$X_2 = L_2\omega + \gamma \tag{114}$$

$$X_3 = -B + \gamma \tag{115}$$

introduce these quantities into (106), and take the limit as γ approaches minus infinity, obtaining

$$X'_1 = X_1 - \frac{\beta^2(L_2\omega - B)}{(R_2 + R_3)^2 + (L_2\omega - B)^2} \tag{116}$$

Replacing B by its value from (69), we have

$$X'_1 = X_1 - \frac{\beta^4 X_1/Z_1^2}{(R_2 + R_3)^2 + \dfrac{\beta^4 X_1^2}{Z_1^4}} \tag{117}$$

The solution of this equation for $X'_1 = 0$ is

either
$$X_1 = 0 \tag{118}$$

or
$$Z_1^2 = \frac{\beta^2 R_1}{R_2 + R_3} \tag{118a}$$

Expressing these results in terms of ratio constants, we have

either
$$J_1 = 0 \tag{119}$$

or
$$\eta_1^2 + J_1^2 = \frac{\tau^2\eta_1}{\eta_2(\rho + 1)} \tag{120}$$

To decide which of these conditions is to be used in a given case, it is only necessary to note that since C_{23} is zero, we have *the case of two circuits* with the secondary circuit made up of the inductance L_2, the capacity C_{30}, and the resistance $R_3 + R_2$. An examination along these lines, making use of Chapters XI and XII, shows that (120) is to be used whenever it is attainable.

When it is not attainable (119) is to be used. The case for (119) is the case of *deficient coupling*, while the case for (120) is the case of *sufficient coupling.*

The smallest value that J_1^2 can have is 0, so (120) is attainable, provided

$$\eta_1^2 \leqq \frac{\tau^2\eta_1}{\eta_2(\rho+1)}, \; i.e., \; \eta_1\eta_2(\rho+1) \leqq \tau^2 \tag{121}$$

whence

$$\rho \leqq \frac{\tau^2}{\eta_1\eta_2} - 1 \tag{122}$$

When (122) *is satisfied, the optimum value of* J_1 *for Resonance Combination* 0 *is given by* (120). *When* (122) *is not satisfied the optimum value of* J_1 *is* (119).

We shall now obtain values of the grand maxima of relative power in this 0-case. Substituting (90) into (103) we have

$$H_0 = \frac{1}{\left\{r\left(\frac{1}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{1}\right)\sqrt{\frac{\eta_2}{\eta_1}} + \frac{\tau^2}{r\sqrt{\rho}}\sqrt{\frac{\eta_1}{\eta_2}}\right\}^2} \tag{123}$$

In case

$$J_1 = 0,$$

r by its definition (83) reduces to η_1, and then (123) becomes

$$[H_{\max}]_0 = \frac{1}{\frac{\eta_1\eta_2}{\rho}\left\{1 + \rho + \frac{\tau^2}{\eta_1\eta_2}\right\}^2}, \text{ for } J_1 = 0 \tag{124}$$

In case

$$J_1^2 + \eta^2 = r^2 = \frac{\tau^2\eta_1}{\eta_2(\rho+1)},$$

if we first square the brace of the denominator of (123), and then replace r^2, we have

$$[H_{\max.}]_0 = \frac{1}{4\tau^2\left(1 + \frac{1}{\rho}\right)}, \text{ for } J_1^2 = \frac{\tau^2\eta_1}{\eta_2(\rho+1)} - \eta^2 \tag{125}$$

We may sum up these results as follows: With Resonance Combination 0, *if* ρ $(=R_3/R_2)$ *satisfies the inequality* (122), *the optimum value of* J_1 *is given by* (120). *The value of this power is given by* (125). *If, on the other hand,* ρ *does not satisfy the inequality* (122), *the optimum value of* J_1 *is given by* (119), *and the value of the relative power at this maximum is given by* (124).

241. Investigation of the Grand Maxima of Power with Respect to Resonance Combination A.—If we substitute (93) into (103) we shall have the Relative Power

$$H_A = \frac{1}{4\left(\frac{\eta_2 r^2}{\eta_1} + \tau^2\right)} \tag{126}$$

In this equation, r^2, which is defined by (83) contains the reactance constant J_1, and it is seen by inspection that the adjustment of J_1 that makes (126) a maximum is

$$J_1 = 0,$$

and that the value of H_A at this adjustment is

$$[H_{\max}]_A = \frac{1}{4(\eta_1\eta_2 + \tau^2)} \tag{127}$$

The condition under which this maximum is attainable is had by setting $J_1 = 0$ in the criterion inequality given immediately preceding equation (91) in the "Summary and Key in Terms of Ratio Quantities," Art. **234.**

This operation gives

$$1 + \frac{\tau^2}{\eta_1\eta_2} \leqq \rho \leqq 1 + \frac{\tau^2}{\eta_1\eta_2} + \frac{\eta_1}{\eta_2(\eta_1\eta_2 + \tau^2)} \tag{128}$$

The Resonance Combination A is attainable if the inequality (128) *is satisfied by* ρ $(= R_3/R_2)$, *and the optimum value of* J_1 *is* $J_1 =$ 0. *The value of the relative power at this adjustment is given by* (127).

242. Investigation of the Grand Maxima of Power with Respect to Resonance Combination B.—The substitution of (95) into (103) gives

$$H_B = \frac{1}{\frac{r^2}{\rho\eta_1\eta_2}\left\{a + \frac{\rho(\eta_2{}^2 + \tau^2\eta_1\eta_2/r^2)}{a}\right\}^2} \tag{129}$$

Indicating the denominator of this expression by D, we shall have in expanded form

$$D = \frac{1}{\rho\eta_1\eta_2}\left\{r^2a^2 + 2r^2\rho\eta_2{}^2 + 2\rho\tau^2\eta_1\eta_2 + \frac{r^4\rho^2\eta_2{}^4 + 2r^2\rho^2\tau^2\eta_1\eta_2{}^3 + \rho^2\tau^4\eta_1{}^2\eta_2{}^2}{r^2a^2}\right\} \tag{130}$$

It is required to find the value of J_1 that will make D a minimum. Setting equal to zero the derivative of D with respect to J_1, we have

$$0 = \left\{1 - \frac{(r^2\rho\eta_2{}^2 + \rho\tau^2\eta_1\eta_2)^2}{(r^2a^2)^2}\right\} \frac{\partial(r^2a^2)}{\partial J_1} + 2\rho\eta_2{}^2\left\{1 + \frac{r^2\rho\eta_2{}^2 + \rho\tau^2\eta_1\eta_2}{r^2a^2}\right\}\frac{\partial r^2}{\partial J_1} \qquad (131)$$

The second brace is a common factor that cannot vanish. It may be divided out. Doing this and replacing the derivatives obtainable from (82) and (83), we have

$$0 = \left\{1 - \frac{r^2\rho\eta_2{}^2 + \rho\tau^2\eta_1\eta_2}{r^2a^2}\right\}\{(1 + \eta_2{}^2)J_1 - \tau^2\} + 2\rho\eta_2{}^2J_1 \qquad (132)$$

Clearing this of fractions, we obtain

$$\begin{aligned} 0 = r^2a^2\{(1 + \eta_2{}^2)J_1 - \tau^2 + 2\rho\eta_2{}^2J_1\} - \\ \}(1 + \eta_2{}^2)J_1 - \tau^2\}\ \{r^2\rho\eta_2{}^2 + \rho\tau^2\eta_1\eta_2\} \end{aligned} \qquad (133)$$

Let us write out the value of r^2a^2 by using (83) and (84), obtaining

$$r^2a^2 = (1 + \eta_2{}^2)\ (\eta_1{}^2 + J_1{}^2) + \tau^4 + 2\tau^2(\eta_1\eta_2 - J_1) \qquad (134)$$

Note also, by (83)

$$r^2 = \eta_1{}^2 + J_1{}^2 \qquad (135)$$

The values given in (134) and (135) substituted into (133) gives

$$0 = P_3J_1{}^3 + P_2J_1{}^2 + P_1J_1 + P_0 \qquad (136)$$

where

$$\left.\begin{aligned} P_3 &= (1 + \eta_2{}^2 + \rho\eta_2{}^2)\ (1 + \eta_2{}^2) \\ P_2 &= (1 + \eta_2{}^2 + \rho\eta_2{}^2)\ (-3\tau^2) \\ P_1 &= \{1 + \eta_2{}^2 + 2\rho\eta_2{}^2\}\ \{(1 + \eta_2{}^2)\eta_1{}^2 + \tau^4 + \\ &\quad 2\tau^2\eta_1\eta_2\} + 4\tau^4 - \rho\eta_1\eta_2\{1 + \eta_2{}^2\}\ \{\tau^2 + \eta_1\eta_2\} \\ P_0 &= -\tau^2(\tau^2 + \eta_1\eta_2)^2 + \tau^2\eta_1\eta_2\rho\ (\tau^2 + \eta_1\eta_2) - \tau^2\eta_1{}^2 \end{aligned}\right\} \qquad (137)$$

When the quantities on the right-hand side of equations (137) are numerically known, the cubic equation (136) may be solved by "trial and error" or by other known methods of solving a cubic equation with numerical coefficients.

The cubic equation (136) *gives the value of* J_1 *at which occurs a grand maximum (or a minimum) of relative power with respect to the Resonance Combination B.*

From the solutions obtained for the cubic in any numerical case, one must decide by a separate investigation which of the solutions give maxima and which minima of power, and one must determine the value of the grand maximum of power by substituting the resulting value of J_1 into (129), *in which a and r are functions of J_1 as defined in* (83) *and* (84).

We shall follow the exact treatment here given by approximations that are useful in important cases.

243. Approximate Treatment of the Grand Maximum of Power with Respect to Resonance Combination B.—Instead of employing the cubic equation (136) to determine the value of J_1 at which occurs a grand maximum of relative power with respect to Resonance Combination B, we may obtain an approximate result as follows:

In the value of r^2a^2 given in (134) let

$$\left.\begin{aligned} \eta_1^2 &<< 1 \\ \eta_1^2 &<< J_1^2 + \tau^4 \\ \text{and} \qquad 2\tau^2\eta_1\eta_2 &<< J_1^2 + \tau^4 \end{aligned}\right\} \tag{138}$$

then by (134)

$$r^2a^2 = (J_1 - \tau^2)^2 \text{ approximately} \tag{139}$$

In advance of a determination of J_1^2 we know that it is positive, so that conditions (138) are satisfied, provided

$$\eta_1^2 << 1, \ \eta_1^2 << \tau^2, \ \eta_1\eta_2 << \tau^2/2 \tag{140}$$

The inequalities (140) *give conditions under which the approximation* (139) *is applicable. It may be that* (140) *is more restrictive as to η_1 and η_2 than is necessary. From* (139) *this is seen to be the case when J_1^2 is sufficiently different from zero to add appreciably to τ^4 on the right-hand side of* (138).

If now we substitute (139) into (133), and neglect further η_2^2 where it occurs in comparison with unity, we obtain

$$0 = \{J_1 - \tau^2\}^2\{J_1 - \tau^2 + 2\rho\eta_2^2J_1\} \\ - \{J_1 - \tau^2\}\{r^2\rho\eta_2^2 + \rho\tau^2\eta_1\eta_2\} \tag{141}$$

which, factored and with r^2 replaced by its value from (135), gives

$$\left.\begin{aligned} 0 &= \{J_1 - \tau^2\}\{(J_1 - \tau^2)[(1 + 2\rho\eta_2^2)J_1 - \tau^2] - \\ &\qquad (\eta_1^2 + J_1^2)\rho\eta_2^2 - \rho\tau^2\eta_1\eta_2\} \\ &= \{J_1 - \tau^2\}\left\{J_1^2 - 2\tau^2J_1 + \frac{\tau^4 - \rho\eta_1^2\eta_2^2 - \rho\tau^2\eta_1\eta_2}{1 + \rho\eta_2^2}\right\} \end{aligned}\right\} \tag{142}$$

Setting these two factors separately equal to zero, and solving for J_1, we obtain

either $$J_1 = \tau^2 \tag{143}$$

or $$J_1 = \tau^2 \pm \sqrt{\frac{(\tau^4\eta_2^2 + \eta_1^2\eta_2^2 + \tau^2\eta_1\eta_2)\rho}{1 + \rho\eta_2^2}} \tag{144}$$

Equations (143) *and* (144) *give approximate values of* J_1 *at which grand maxima* (*or minima*) *of power occur with respect to the Resonance Combination B.*

We shall next show that (143) *is the condition for a minimum, and that* (144) *is the approximate value for a grand maximum. This result will be incident to a determination of the magnitude of the Power-maximum.*

244. The Magnitude of the Relative Power with Respect to Resonance Combination B.—Before we introduced any approximations into the examination of the Resonance Combination B, we found that J_1, in order to give a grand maximum of power, must satisfy (133), which was subsequently put into the form (136). We may, therefore, utilize (133) as far as possible to simplify the power equation (129). Concerning ourselves particularly with the denominator of (129), which is given in (130) we may write (130) in the form

$$D = \frac{1}{\rho\eta_1\eta_2}\left(r^2a^2 + 2x + \frac{x^2}{r^2a^2}\right) \tag{145}$$

where, as an abbreviation

$$x = \rho(r^2\eta_2^2 + \tau^2\eta_1\eta_2) \tag{146}$$

We may factor (145) so as to give

$$D = \frac{1}{\rho\eta_1\eta_2}\,\frac{(r^2a^2 + x)^2}{r^2a^2} \tag{147}$$

Equation (147) *gives the denominator of the power equation* (129), *before any resonance conditions regarding* J_1 *are introduced.*

We shall now transform (133), by introducing the abbreviation x. This gives

$$x = r^2a^2\left\{1 + \frac{2\rho\eta_2^2 J_1}{(1 + \eta_2^2)J_1 - \tau^2}\right\} \tag{148}$$

Equation (148) *is the equivalent of* (133), *and is the relation that* J_1 *must satisfy to give a maximum* (*or minimum*) *value of power* H_B.

Substituting (148) into (147) in such a way as to eliminate r^2a^2, we obtain

$$D = \frac{4x}{\rho\eta_1\eta_2} \frac{\left\{1 + \dfrac{\rho\eta_2{}^2 J_1}{(1 + \eta_2{}^2)J_1 - \tau^2}\right\}^2}{1 + \dfrac{2\rho\eta_2{}^2 J_1}{(1 + \eta_2{}^2)J_1 - \tau^2}} \tag{149}$$

Simplifying this expression, replacing x by its value from (146), with r replaced by (135), and introducing the resulting value of D into (129), we obtain

$$[H_{\text{max.}}]_B = \frac{\eta_1[(1 + \eta_2{}^2)J_1 - \tau^2 + 2\rho\eta_2{}^2 J_1][(1 + \eta_2{}^2)J_1 - \tau^2]}{4[(\eta_1{}^2 + J_1{}^2)\eta_2 + \tau^2\eta_1][(1 + \eta_2{}^2)J_1 - \tau^2 + \rho\eta_2{}^2 J_1]^2} \tag{150}$$

Into equation (150) *the values of* J_1 *given by the cubic equation* (136) *must be introduced. The resulting values will be either maxima or minima of H. Only the maxima are to be selected, and are to be used with the other adjustments incident to the Resonance Combination B.*

245. Approximate Magnitude of Relative Power with Respect to Resonance Combination B.—In equations (143) and (144) we have found approximate values of J_1 that give either a maximum or a minimum of power with respect to Resonance Combination B. We may write (144)

$$J_1 = \tau^2 + \alpha \quad \text{(approximately)} \tag{151}$$

where

$$\alpha = \pm\sqrt{\frac{(\tau^4\eta_2{}^2 + \eta_1{}^2\eta_2{}^2 + \tau^2\eta_1\eta_2)\rho}{1 + \rho\eta_2{}^2}} \tag{152}$$

Introducing (151) into (150), with neglect of $\eta_2{}^2$ in comparison with unity, we obtain

$$[H_{\text{max}}]_B = \frac{\alpha\eta_1\{(1 + 2\rho\eta_2{}^2)\alpha + 2\rho\eta_2{}^2\tau^2\}}{\{(\tau^2 + \alpha)^2\eta_2 + \eta_1{}^2\eta_2 + \tau^2\eta_1\}\{\alpha(1 + \rho\eta_2{}^2) + \rho\eta_2{}^2\tau^2\}^2} \tag{153}$$

It is seen from this equation that $J_1 = \tau^2$ gives a minimum of power, for this is equivalent to making $\alpha = 0$ in (151), and gives relative power zero, to the accuracy of the approximations employed in deducing (153).

Using the approximate resonance value of J_1 *given in* (151), *equation* (153) *gives an approximate value of the grand maximum of power with Resonance Combination B.*

We shall now make a collection of the several optimum Resonance Combinations, which are the Resonance Combinations

L, 0, A, and B, with, however, their corresponding optimum values of J_1 also taken into account.

246. Collection of Optimum Resonance Combinations.—In the "Summary and Key in Terms of Ratio Constants," Art. **234**, we have given a list of Resonance Combinations designated L, 0, A, and B. In the pages following the Key we have determined the value of J_1 that will give grand maximum expenditure of power in the resistance R_3 for each of the Resonance Combinations. When J_1 is thus made optimum with the several Resonance Combinations, we shall designate the combinations Optimum Resonance Combinations L, 0, A, and B.

Incidentally there have appeared two Optimum Resonance Combinations 0, which we shall refer to as 0_0 and 0_1.

In stating the various combinations, we shall need to introduce the optimum value of J_1 into the statement of the optimum adjustment of the Circuits I and II whenever these adjustments are functions of J_1.

We shall also collect along with the Optimum Resonance Combinations the values of the Grand Maximum of Relative Power for those combinations.

We shall later, where possible, lay down rules as to which of the optimum combinations is to be used for any given relation among the resistances of the three circuits.

L. The optimum resonance combination L is

$$C_{23} = 0, \quad C_{30} = \infty, \quad J_1 = \frac{\tau^2}{\eta_2^2(1+\rho)^2 + 1} \tag{154}$$

and the relative power at this adjustment is

$$[H_{\max}]_L = \frac{1}{\dfrac{\eta_1\theta}{\eta_2\rho} + \dfrac{\tau^4}{\rho\eta_1\eta_2} + 2\tau^2\left(\dfrac{1}{\rho} + 1\right) - \dfrac{\tau^4}{\rho\eta_1\eta_2\theta}} \tag{155}$$

where

$$\theta = 1 + \eta_2^2(1+\rho)^2 \tag{156}$$

$$\rho = R_3/R_2 \tag{157}$$

0_0. The optimum resonance combination 0_0 is

$$C_{23} = 0, \quad \frac{\Lambda_3^3}{\lambda^2} = 1, \quad J_1 = 0 \tag{158}$$

and the relative power at this adjustment is

$$[H_{\max}]_{0_0} = \frac{1}{\frac{\eta_1\eta_2}{\rho}\left(1 + \rho + \frac{\tau^2}{\eta_1\eta_2}\right)^2} \tag{159}$$

This combination is not to be used when

$$\rho < \frac{\tau^2}{\eta_1\eta_2} - 1 \tag{160}$$

0_1. The optimum resonance combination 0_1 is

$$\left.\begin{aligned} C_{23} &= 0 \\ \frac{\Lambda_3^2}{\lambda^2} &= \frac{1}{1 \pm \frac{\eta_2(\rho+1)}{\eta_1}\sqrt{\frac{\tau^2\eta_1}{\eta_2(\rho+1)} - \eta_1^2}} \\ J_1^2 &= \frac{\tau^2\eta_1}{\eta_2(\rho+1)} - \eta_1^2 \end{aligned}\right\} \tag{161}$$

and the relative power at this adjustment is

$$[H_{\max.}]_{0_1} = \frac{1}{4\tau^2\left(1 + \frac{1}{\rho}\right)} \tag{162}$$

This combination can be attained only provided

$$\rho \leqq \frac{\tau^2}{\eta_1\eta_2} - 1 \tag{163}$$

A. The optimum resonance combination A is

$$\left.\begin{aligned} \frac{\Lambda_2^2}{\lambda^2} &= \\ &\frac{1 - \sqrt{\frac{1}{\rho}\left\{1 + \frac{\tau^2}{\eta_1\eta_2}\right\}\left\{1 + \left(\frac{\eta_1\eta_2 + \tau^2}{\eta_1}\right)^2 - \frac{\rho\eta_2}{\eta_1}(\eta_1\eta_2 + \tau^2)\right\}}}{1 + \left(\frac{\eta_1\eta_2 + \tau^2}{\eta_1}\right)^2} \\ \frac{\Lambda_3^2}{\lambda^2} &= \sqrt{\frac{1 + \frac{\tau^2}{\eta_1\eta_2}}{\rho\left\{1 + \left(\frac{\eta_1\eta_2 + \tau^2}{\eta_1}\right)^2 - \frac{\rho\eta_2}{\eta_1}(\eta_1\eta_2 + \tau^2)\right\}}} \\ J_1 &= 0 \end{aligned}\right\} \tag{164}$$

and the relative power at this adjustment is

$$[H_{\max}]_A = \frac{1}{4(\eta_1\eta_2 + \tau^2)} \tag{165}$$

This combination can be attained only provided

$$1 + \frac{\tau^2}{\eta_1\eta_2} \leqq \rho \leqq 1 + \frac{\tau^2}{\eta_1\eta_2} + \frac{\eta_1}{\eta_2(\eta_1\eta_2 + \tau^2)} \tag{166}$$

B. The optimum resonance combination B is accurately

$$\frac{\Lambda_2^2}{\lambda^2} = \frac{b}{a^2}, \; C_{30} = \infty \tag{167}$$

with J_1 a root of the cubic equation

$$P_0 + P_1J_1 + P_2J_1^2 + P_3J_1^3 = 0 \tag{168}$$

with the values of P_0, P_1, P_2, and P_3 given in equation (137)

To obtain the values of $[H_{\text{max.}}]_B$, the values obtained for J_1 are to be substituted into (150), and then the minimum values, if any appear, are to be discarded.

To obtain the adjustment appropriate to Circuit II, the values of J_1 that give maximum values of H are to be introduced into b and a^2 of (167), in accordance with the definitions of b and a^2 given in (82) and (83).

B. Approximate:

When

$$\eta_1^2 << 1, \quad \eta_1^2 << \tau^2, \text{ and } \eta_1\eta_2 << \tau^2/2 \tag{169}$$

the optimum resonance combination B is approximately

$$\left.\begin{aligned} \frac{\Lambda_2^2}{\lambda^2} &= \frac{\eta_1^2 + \tau^2\alpha + \alpha^2}{\eta_2^2(\eta_1^2 + 2\tau^2\alpha + \alpha^2) + (\eta_1 + \eta_2\tau^2)^2 + \alpha^2} \\ C_{30} &= \infty \\ J_1 &= \tau^2 + \alpha \end{aligned}\right\} \tag{170}$$

and the relative power at this adjustment is

$$[H_{\max}]_B = \frac{\alpha\eta_1\{(1 + 2\rho\eta_2^2)\alpha + 2\rho\eta_2^2\tau^2\}}{\{(\tau^2 + \alpha)^2\eta_2 + \eta_1^2\eta_2 + \tau^2\eta_1\}\{\alpha(1 + \rho\eta_2^2) + \rho\eta_2^2\tau^2\}^2} \tag{171}$$

where

$$\alpha = \pm\sqrt{\frac{[\text{illegible}]}{1 + \rho\eta_2^2}} \tag{172}$$

247. Comparison of the Grand Maxima of Power for the Several Optimum Resonance Combinations.—By comparing the values of the relative power ($H_{\max}$) for the several combinations, we are able to decide which combination gives the greatest relative power for any given value of ρ ($= R_3/R_2$). The results are given in Table V.

Table V.—Proper Optimum Resonance Combinations for Different Values of R_3/R_2

Value of ρ $(=R_3/R_2)$	Use optimum resonance combination designated
$0 \leqq \rho \leqq -1 + \dfrac{\tau^2}{\eta_1\eta_2}$	0_1
$-1 + \dfrac{\tau^2}{\eta_1\eta_2} \leqq \rho \leqq 1 + \dfrac{\tau^2}{\eta_1\eta_2}$	0_0
$1 + \dfrac{\tau^2}{\eta_1\eta_2} \leqq \rho \leqq 1 + \dfrac{\tau^2}{\eta_1\eta_2} + \dfrac{\eta_1}{\eta_2(\eta_1\eta_2 + \tau^2)}$	A
$1 + \dfrac{\tau^2}{\eta_1\eta_2} + \dfrac{\eta_1}{\eta_2(\eta_1\eta_2 + \tau^2)} \leqq \rho \leqq \infty$	B

Table V was obtained (by steps not here given) by noting first that the optimum combination A could be attained only when ρ was within the limits assigned in (166), which are the limits given in the third line of the table. By subtraction of the denominator in the expression for Power in the A-case from the corresponding denominator in the 0_0-case, it was found that the denominator in the A-case was always the smaller, so that combination A, when attainable, gives more power than combination 0_0. It was next noted that combinations A and 0_1 are never attainable together, since the upper value of ρ for which 0_1 is attainable is

$$\rho \leqq -1 + \frac{\tau^2}{\eta_1\eta_2},$$

by (163).

It was then shown by subtracting power-denominators that combination 0_1 always gives more power than combination 0_0 so that 0_0 is to be used only when 0_1 and A are both unattainable. This range is given in the second line of Table V.

We are left in doubt up to here whether B or L should replace 0_1, 0_0, or A in the ranges corresponding to the first three lines of Table V. A subtraction of the denominators in the case 0_1, 0_0, and A successively from the denominator in the case of combination L, shows that the combination L is not superior to any of the other combinations within the ranges given in the table.

As to combination B, it is in such a form that there is difficulty

in determining by direct subtraction whether or not the power for the B combination is greater than that for the other combinations. We find, however, that the B combination gives $J_1 = 0$, when

$$\rho = 1 + \frac{\tau^2}{\eta_1\eta_2} + \frac{\eta_1}{\eta_2(\tau^2 + \eta_1\eta_2)} \tag{173}$$

which is the adjustment of J_1 for the A-combination. Also at this adjustment the value of the power for the B-combination agrees with the value of the power for the A-combination.

The inference from this is that the B-combination has application to values of ρ greater than the limit given in (173), and this inference is entered in Table V.

IV. COMPUTATION OF OPTIMUM ADJUSTMENTS AND GRAND MAXIMA OF POWER IN A SPECIAL CASE

248. The Optimum Adjustments of the Primary Circuit (Circuit I) in a Special Case, with $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$.—In the "Collection of Optimum Resonance Combinations," Art. **246,** there are given formula for computing the adjustments of the constants of the circuits to produce maxima of relative power in Circuit III. We shall here give the adjustments of Circuit I, in the form of values of J_1, where

$$J_1 = 1 - \frac{\lambda_2}{\Lambda_1^2} \tag{174}$$

where

λ = the wavelength of the impressed e.m.f.
Λ_1 = the undamped wavelength of Circuit I.

The optimum value of J_1, which is the quantity computed will be sometimes designated $J_{1\,\mathrm{opt}}$.

With the values of τ, η_1, and η_2 given in the caption, I have computed the values of $J_{1\,\mathrm{opt}}$ for various values of the ratio R_3/R_2, where R_3 is the resistance of Circuit III containing the detector, and R_2 is the resistance of the Circuit II. The values employed for R_3/R_2 extend from 1 to 100,000.

Fig. 6 gives the values of $J_{1\,\mathrm{opt}}$ at which the grand maxima of power occur for values of R_3/R_2 up to 700. The different parts of the curves are labelled to accord with the optimum resonance combinations L, 0, A, and B employed in their computation. The actual amount of relative power for these adjustments are given in the next section.

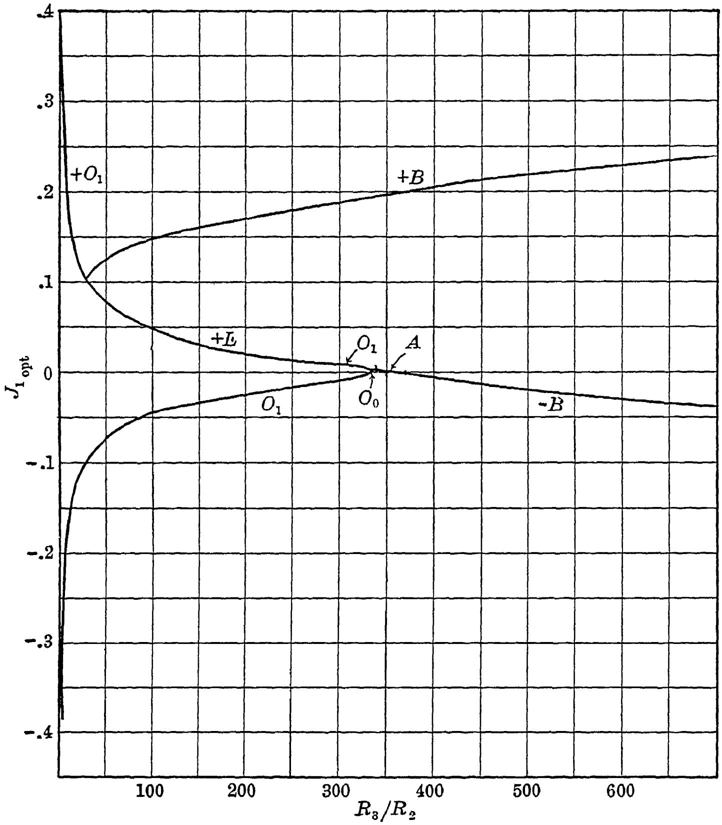

FIG. 6.—Values of $J_{1\mathrm{opt}}$ for various values of R_3/R_2. The letters attached to the curves indicate the resonance relations employed.

Table VI.—Optimum Values of J_1 for Large Values of R_3/R_2, with the Given Values of τ, η_1, and η_2

R_3/R_2	J_1 opt.	
7,000	0 457	−0 257
8,000	0 471	−0 271
9,000	0 483	−0 283
10,000	0 493	−0 293
15,000	0 531	−0 331
20,000	0 554	−0 354
50,000	0 609	−0 409
100,000	0 632	−0 432
∞	0 657	−0 457

Continuing the examination merely of the optimum values of J_1, Fig. 7 contains the same curves as Fig. 6, with, however, a different scale for R_3/R_2, and an extension of the results to values of R_3/R_2 up to 7000.

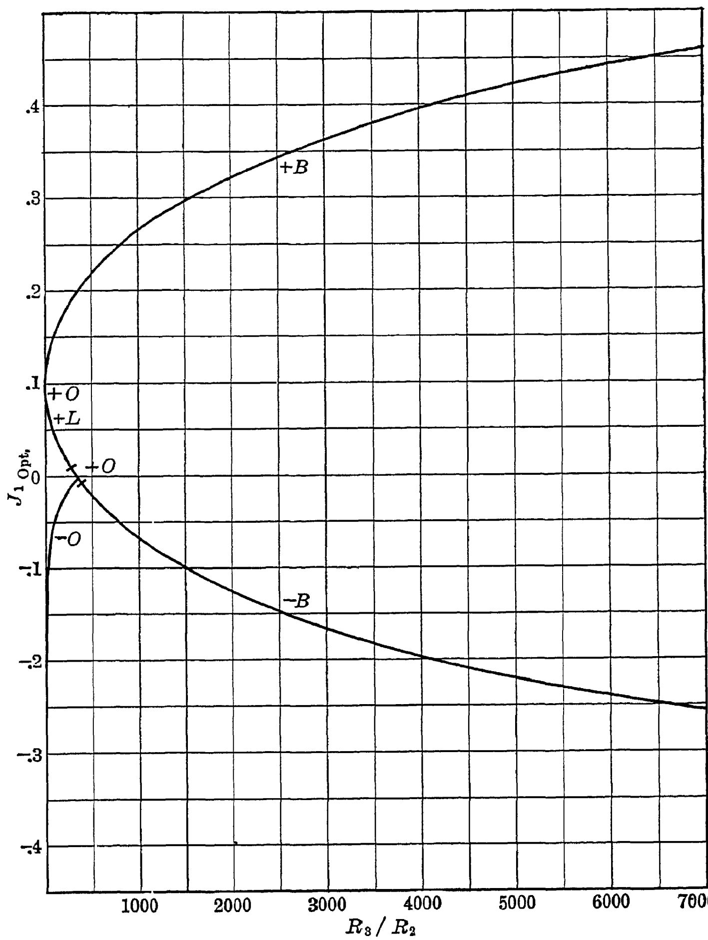

FIG. 7.—Extension of Fig. 6.

Beyond the ratio of resistances R_3/R_2 equal to 7000, curves are not given, but the computed results are contained in Table VI.

In all calculations involving Combination B the approximate equations (170) and (171) were employed.

249. Magnitudes of the Grand Maxima of Power for Various Values of R_3/R_2. Given $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$.—Using the formulas collected in equations (154) to (172) and employing resistance ratios from 1 to 50,000, values of the relative power expended in the detector were computed in the special case of $\tau^2 = 0.1$, $\eta_1 = 0.03$, $\eta_2 = 0.01$, with the results given in Figs. 8 and 9.

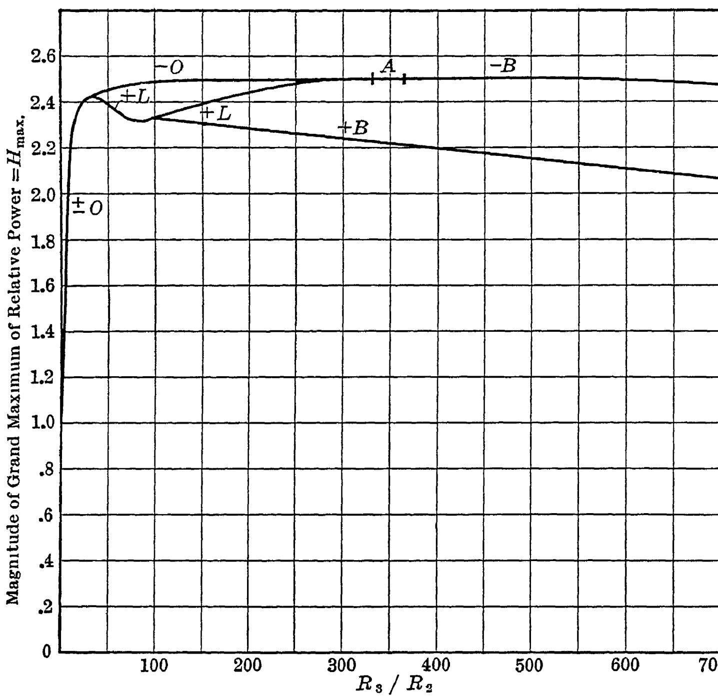

FIG 8—Grand maximum of relative power vs. R_3/R_2.

Fig. 8 is for the range of R_3/R_2 from 0 to 700. Fig. 9 is for the range of R_3/R_2 from zero to 14,000.

The extension of the range to 50,000 is given in Table VII.

An examination of the curves of Figs. 8 and 9, and Table VII shows that with this particular set of constants, τ, η_1, and η_2, the detector in which the greatest power is developed has a resistance between 150 and 600 times the resistance of the secondary inductance coil, and that the optimum adjustment of the circuits comes under the cases of Optimum Resonance Combinations A and B.

As the resistance increases beyond 600 times the resistance

of the secondary coil, the power expended in the detector decreases. With a different coefficient of coupling and different values of η_1 and η_2 this optimum range of resistances for the detector is different.

The problem is too diversified to permit of exhaustive numerical examination.

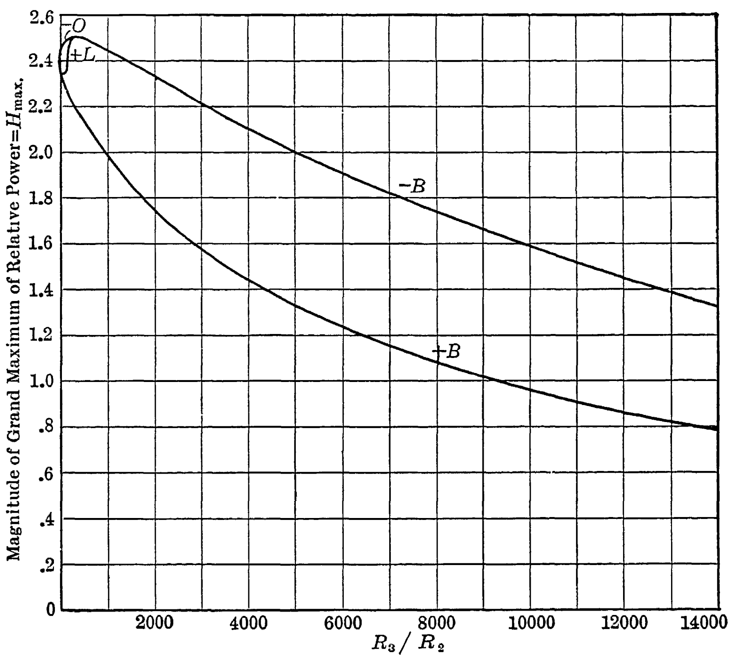

FIG. 9—Extension of Fig 8

Table VII.—Rel. Power in Detector at Opt. Adjustments for Large Values of R_3/R_2

R_3/R_2	Relative power at	
	Positive maximum	Negative maximum
7,000	1 15	1 82
10,000	0 97	1.60
20,000	0 61	1 11
50,000	0 30	0 58

In these Fig. 8 and 9 and in Table VII a maximum of relative power is called a *positive maximum,* or a *negative maximum* according as a *positive* or a *negative* value of J_1 is used in its computation.

V. SOME GENERAL CONCLUSIONS

250. Form of Circuits.—The discussion in this chapter pertains to circuits of the form of Figs. 1 and 2, Art. **220,** in which the detector and a stoppage condenser C_{30} are shunted about the secondary condenser C_{23}. The detector may have any resistance R_3 whatever, and none of the resistances of the various circuits are neglected.

251. When Should C_{23} be Zero?—The question arises as to when is it advisable to have a condenser at C_{23}, and when do the resonance devices of the previous Chapters XI and XII without such a condenser give larger secondary current and larger power development in the detector? The answer is found in a consideration of equations (158) and (161) and of Table V. It is seen in (158) and (161) that $C_{23} = 0$ for combinations 0_0 and 0_1, and in Table V it is seen that these combinations give grand maxima of power whenever

$$\frac{R_3}{R_2} \leqq +1+\frac{\tau^2}{\eta_1\eta_2} \tag{175}$$

Equation (175) *gives the condition under which the tuning of Chapters XI and XII with the secondary coil, the detector, and a variable condenser in series* (*without the C_{23} of the present chapter*), *will give more power in the detector than any adjustment with the use of C_{23}* (*note, C_2 of Chapter X is the C_{30} of present Chapter*).

252. What is the Best Value of the Stoppage Condenser C_{30} for Detectors of High Resistance.—For detectors of sufficiently high resistance to make

$$R_3/R_2 \geqq 1+\frac{\tau^2}{\eta_1\eta_2}+\frac{\eta_1}{\eta_2(\eta_1\eta_2+\tau^2)} \tag{176}$$

the optimum resonance combination is combination B, and in this case the value of the stoppage condenser C_{30} is infinite.

We have then the interesting result that, except for possible requirements of the telephone receiver used as an indicating instrument, the stoppage condenser should be infinite whenever the detector resistance is sufficiently large to satisfy (176).

In our numerical example, (176) becomes

$$R_3/R_2 \geqq 364.3.$$

CHAPTER XVI

ELECTRICAL SYSTEMS OF RECURRENT SIMILAR SECTIONS. ARTIFICIAL LINES. ELECTRICAL FILTERS

253. Utility.—The study of the electrical transmission characteristics of various systems of circuits that consist of recurrent sections in the form of a chain is highly interesting and important.

Circuits of recurrent sections are employed as artificial telephone and telegraph lines.[1] By properly choosing the sections a line similating telephone and telegraph lines or cables may be constructed and employed in electrical experiments in the place of the actual lines.

Circuits of recurrent sections may also be employed as electrical filters[2] for eliminating disturbances from telephone and telegraph circuits. It is believed that such filters may come to have a wide application to the elimination of disturbances from radiotelegraphic receiving stations. Such filters have also interesting applications to bridge measurements and other laboratory operations, in which it is desired to eliminate harmonics and other disturbances.

Further, by properly choosing the constants of the sections the electrical artificial line may be employed to introduce predetermined time retardation of electric currents in a way that gives time retardation practically independent of the frequency

[1] An artificial line with resistances in series and condensers in shunt was patented by Varley in 1862, *British Patent* No. 3453. A similar line but with uniformly distributed capacity and resistance was patented by Taylor and Muirhead, *British Patent* No. 684, of 1875. A line with uniformly distributed inductance, resistance, and capacity was made and described by Pupin, *Trans Am. Inst. of El. Engineers*, 16, pp 93–142, 1899. Another form of uniformly distributed artificial line was constructed and described by Cunningham, *Trans. Am. Inst. of El. Engineers*, 30, pp. 245–256, 1911.

For further references and for an extended treatment of the subject see a recent book by Kennelly, "Artificial Electrical Lines," McGraw-Hill, 1917, from which the above references are taken.

[2] G. M. B. Shepherd, "Note on High-frequency Wave Filters," *The Electrician*, 71, pp. 399–401, 1913. G. A. Campbell, U. S. Patent No. 1227113, 1917.

over wide ranges of frequency. This has been utilized by the author[1] in an *electrical compensator* employed in determining the direction of sources of sound, particularly under water, in submarine boat detection and in submarine signalling. Similar devices are applicable to direction-finding by electric waves and to the elimination of interference in radiotelegraphy by directive receiving.

The principles to be developed in the study of these systems of recurrent sections will serve to show their general application, and will serve also as an introduction to the study of electric waves on wires, to be treated in the next later chapter.

I. GENERAL SYSTEM OF EQUAL SECTIONS

254. General Type of Circuits. Notation.—The discussion will be limited to a system of recurrent sections that are all equal, except at the terminals of the system.

A system of this character, but with considerable generality as to the nature of the sections, is shown in Fig. 1.

The complex impedances z_0, z_1, z_2, and z_T may each consist of any combination of capacities, inductances, and resistances.

Each of the complex impedances z_2 is common to two circuits and is of the nature of a mutual impedance

The impedances z_1 are not common to two circuits, but for the sake of generality there is assumed a mutual inductance[2] between the elements z_1 of each pair of adjacent loops, but no mutual inductance between loops not adjacent.

The complex impedances z_0 and z_T are the impedances of the terminal apparatus at the two ends of the system.

What may be called the line proper, exclusive of the terminal impedances, ends in a half section $z_1/2$ at each end.

255. General Equations.—We shall designate the complex current through the non-common elements of the successive loops as $i_0, i_1, i_2, i_3, \ldots i_{n-1}, i_n$. These currents are supposed to be positive when in the direction of the arrows marked i_0, i_1, etc.

The current i_0 flows through the terminal impedance z_0 at

[1] Description in U. S. Navy Archives and in pending U. S. patent application.

[2] This mutual inductance may be made zero to suit cases in which no such mutual inductance exists.

the *input end*, and the current i_n through the terminal impedance z_T at the *output end*.

We shall treat the problem for only the steady-state[1] condition, under the action of a sinusoidal impressed e.m.f.

If we let the impressed e.m.f. be replaced by an exponential expression

$$e = E\epsilon^{j\omega t},$$

and if we let

$$\left.\begin{aligned} i_0 &= A_0\epsilon^{j\omega t} \\ i_1 &= A_1\epsilon^{j\omega t} \\ i_2 &= A_2\epsilon^{j\omega t} \\ &\cdots\cdots\cdots \\ i_n &= A_n\epsilon^{j\omega t} \end{aligned}\right\} \tag{1}$$

it is seen, as in Chapter XIII, that the complex amplitudes of current A_1, A_2, . . . A_n will be required to satisfy the following algebraic equations obtained from Kirchhoff's *e.m.f.* law:

$$\left.\begin{aligned} E &= \qquad\qquad z'A_0 + bA_1 \\ 0 &= bA_0 + zA_1 \;+ bA_2 \\ 0 &= bA_1 + zA_2 \;+ bA_3 \\ 0 &= bA_2 + zA_3 \;+ bA_4 \\ &\cdots\cdots\cdots\cdots \\ 0 &= bA_{n-1} + z''A_n \end{aligned}\right\} \tag{2}$$

where, as abbreviations,

$$\left.\begin{aligned} z' &= z_0 + z_2 + z_1/2 \\ z'' &= z_T + z_2 + z_1/2 \\ z &= z_1 + 2z_2 \\ b &= Mj\omega - z_2 \end{aligned}\right\} \tag{3}$$

Method of Making All of the Equations (2) Symmetrical. It will be noted that all of the equations (2) may be made symmetrical if we write

$$A_{-1} = \frac{(z' - z)A_0 - E}{b} \tag{4}$$

and

$$A_{n+1} = \frac{(z'' - z)A_n}{b} \tag{5}$$

With these definitions of A_{-1} and A_{n+1}, which have no other

[1] A treatment of the transient state is given by J. R. Carson, *Proc. Am. Inst. of El. Engineers*, 38, p. 407, 1919.

meaning than that given them by the equations (4) and (5), we may replace E in the first equation of (2) and z'' in the last equation of (2), obtaining for the whole set (2)

$$\left.\begin{array}{l} 0 = bA_{-1} + zA_0 + bA_1 \\ 0 = bA_0 \quad + zA_1 + bA_2 \\ 0 = bA_1 \quad + zA_2 + bA_3 \\ \dots\dots\dots\dots\dots \\ 0 = bA_{n-1} + zA_n + bA_{n+1} \end{array}\right\} \qquad (6)$$

Each of the equations (6) is now seen to be of the generic form

$$0 = bA_{m-1} + zA_m + bA_{m+1} \qquad (7)$$

Equation (7) is a generic equation showing the relation of the complex current amplitudes in adjacent sections of the system of the form of Fig. 1. This equation in which m is to be given values

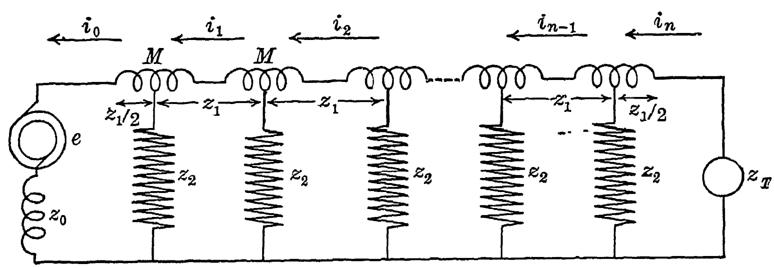

FIG 1 —General system of recurrent equal sections. Complex impedances z_1 and z_2 may be of any character

corresponding to the subscripts in (6), when taken in conjunction with (4) and (5) enables us to obtain a complete solution of the problem of determining the currents in the steady state.

256. Solution for Complex Current Amplitudes.[1]—The equation (7) may be shown to hold for all values of m. For our purpose it will be sufficient to show that it holds for values of m from $m = -1$ to $m = n + 1$. We have already seen in (6) that equation (7) holds for values of m from and including 0 to and including n. To show that the generic equation (7) holds for $m = -1$, let us write down the equation that results from making $m = -1$, obtaining

$$0 = bA_{-2} + zA_{-1} + bA_0.$$

[1] In the theoretical treatment of this subject I have followed the method outlined in G. A. Campbell's U. S Patent No 1227113, 1917.

This amounts merely to a definition of A_{-2} in terms of A_{-1} and A_0, and since A_{-2} has no physical meaning in the problem, we can make this definition, and shall make no further use of it.

In like manner one can satisfy himself that (7) holds also for $m = n + 1$.

We shall now proceed to a solution of (7), which is of a form known as a difference equation. The known method of treating this equation consists in assuming that

$$A_m = G\epsilon^{km} \tag{8}$$

where G is independent of m.

Substituting (8) into (7), and giving to m successively the values $m - 1$, m, and $m + 1$, we obtain

$$0 = G\{b\epsilon^{k(m-1)} + z\epsilon^{km} + b\epsilon^{k(m+1)}\} \tag{9}$$

whence it appears that G is an arbitrary constant.

Now dividing (9) by $G\epsilon^{km}$, we obtain on transposition,

$$\epsilon^k + \epsilon^{-k} = -z/b \tag{10}$$

or otherwise written

$$k = \cosh^{-1}\left\{\frac{-z}{2b}\right\} \tag{11}$$

We may write this result in still a third form by solving (10) as a quadratic in ϵ^k, and this gives

$$\left.\begin{aligned}\epsilon^k &= -\frac{z}{2b} \mp \sqrt{\frac{z^2}{4b^2} - 1}\\ \epsilon^{-k} &= -\frac{z}{2b} \pm \sqrt{\frac{z^2}{4b^2} - 1}\end{aligned}\right\} \tag{12}$$

Let us note also that if k satisfies (10) then $-k$ also satisfies it, since k and $-k$ enter into (10) symmetrically. Therefore we have another solution of (7) in the form

$$A_m = H\epsilon^{-km} \tag{13}$$

where H is also arbitrary and independent of m.

In order to distinguish between k and $-k$, both of which are, in general, complex quantities, we shall specify that k has its real part positive.

Now, since (7) is linear and homogeneous in A_m, the sum of the two solutions is a solution; hence

$$A_m = G\epsilon^{km} + H\epsilon^{-km} \tag{14}$$

where G and H are both arbitrary and independent of each other and of m.

Equation (14), *since it contains two arbitrary constants, is known by the theory of difference equations to be the most general solution of the given difference equation* (7). *In* (14) *k has the value given by* (10), (11), *or* (12).

257. Introduction of Terminal Conditions, and the Determination of the Arbitrary Constants G and H.—To obtain the values of the arbitrary constants G and H, let us substitute (14), with proper value of m, into (4) and (5).

We thus obtain

$$G\epsilon^{-k} + H\epsilon^{k} = \frac{(z' - z)(G + H)}{b} - \frac{E}{b} \tag{15}$$

and

$$G\epsilon^{k(n+1)} + H\epsilon^{-k(n+1)} = \frac{z'' - z}{b}\{G\epsilon^{kn} + H\epsilon^{-kn}\} \tag{16}$$

As abbreviations let us write

$$\frac{z' - z}{b} = x, \text{ whence by (3) } x = \frac{z_0 - z_2 - z_1/2}{b} \tag{17}$$

and

$$\frac{z'' - z}{b} = y, \text{ whence by (3) } y = \frac{z_T - z_2 - z_1/2}{b} \tag{18}$$

By transposition of (15) and (16) and by the employment of (17) and (18), we obtain

$$G(\epsilon^{-k} - x) + H(\epsilon^{k} - x) = -E/b \tag{19}$$

and

$$G\epsilon^{kn}(\epsilon^{k} - y) + H\epsilon^{-kn}(\epsilon^{-k} - y) = 0 \tag{20}$$

As further abbreviations let us write

$$X = -\frac{\epsilon^{-k} - x}{\epsilon^{k} - x} \tag{21}$$

$$Y = -\frac{\epsilon^{-k} - y}{\epsilon^{k} - y} \tag{22}$$

then (20) gives

$$G = HY\epsilon^{-2kn} \tag{23}$$

The substitution of (23) and (21) into (19) gives

$$H = -\frac{E}{b(\epsilon^{k} - x)}\,\frac{1}{1 - \epsilon^{-2kn}XY} \tag{24}$$

and by (23),

$$G = -\frac{E}{b(\epsilon^{k} - x)}\,\frac{Y\epsilon^{-2kn}}{1 - \epsilon^{-2kn}XY} \tag{25}$$

These values of G and H substituted into (14) gives for the complex current amplitude A_m the value

$$A_m = -\frac{E}{b(\epsilon^k - x)} \cdot \frac{\epsilon^{-km} + Y\epsilon^{-k(2n-m)}}{1 - XY\epsilon^{-2kn}} \tag{26}$$

Equation (26) *gives the complex current amplitude* A_m *of the current in the mth section. X and Y are given by* (21) *and* (22), *x is given by* (17); *k, by* (10), (11), *or* (12).

258. Analysis of the Complex Current Amplitude into a Summation, Exhibiting the Effects of Repeated Reflection.—The expression (26) for A_m may be put into a more interesting form by expanding one of the factors as follows:

$$\frac{1}{1 - XY\epsilon^{-2kn}} = 1 + XY\epsilon^{-2kn} + X^2Y^2\epsilon^{-4kn} + X^3Y^3\epsilon^{-6kn} + \quad . \; . \tag{27}$$

Introducing this into (26) we obtain

$$\begin{aligned} A_m = -\frac{E}{b(\epsilon^k - x)} \Big\{ & \epsilon^{-km} + Y\epsilon^{-k(2n-m)} + XY\epsilon^{-k(2n+m)} \\ & + XY^2\epsilon^{-k(4n-m)} + X^2Y^2\epsilon^{-k(4n+m)} \\ & + X^2Y^3\epsilon^{-k(6n-m)} + X^3Y^3\epsilon^{-k(6n+m)} \\ & + \qquad \ldots\ldots\ldots\ldots \Big\} \end{aligned} \tag{28}$$

This is a variant equation for A_m, *the complex current amplitude in the mth section of a line that terminates with the nth section.*

In equation (28) it is to be noticed that the multipliers of $-k$ in the exponents of the successive terms are as follows:

Term	Multiplier in exponent	Interpretation
First	m	= the number of steps to the mth section from e.m.f direct.
Second	$2n - m$	= the number of steps to outer end and back to the mth section
Third	$2n + m$	= the number of steps to outer end, back to beginning end, and then to mth section.
Etc.	etc.	etc.

These several exponential terms are consistent with the view that the first term is due to direct transmission from the source, while the succeeding terms are due to successive reflections of current from the terminals of the line. Each step from section to section, on this theory, multiplies the complex current amplitude by the constant (complex) factor ϵ^{-k}.

To account for the multipliers X and Y applied successively to the terms after the first, it is only necessary to suppose that Y is the complex reflection coefficient of the terminal of the line remote from the e.m.f., and that X is the complex reflection coefficient of the terminal at the e.m.f.

259. Complex Current Amplitude in the mth Section of an Infinite Line or of a Line with Non-reflective Output Impedance. If the total number of sections n is infinite, or if Y is zero, all the exponentials in (28), except the first, disappear, and we have

$$A_m = \frac{E}{\bar{z}} \epsilon^{-km}, \qquad \text{for } n = \infty, \text{ or } Y = 0 \qquad (29)$$

where

$$\bar{z} = -b(\epsilon^k - x) \qquad (30)$$

Equation (29) gives the complex current amplitude A_m in the mth section of a line of an infinite number of sections or of a line whose reflection coefficient at the remote end is $Y = 0$.

260. Input Impedance, or Surge Impedance, of an Infinite Line or a Line with Non-reflective Output Impedance.—The input impedance, or as it is sometimes called the *surge impedance, of a line is the impedance by which an infinite line of the same kind of sections as the given line may be replaced without changing the current in the zeroth section.*

For the purpose of this discussion the impedance z_0 represented as inserted in the zeroth section will be regarded as a part of the impedance of the input apparatus, and not a part of the line itself. In like manner, z_T may be regarded as the impedance of the output apparatus attached to the line.

For an infinite line or a line of zero output reflection coefficient, we can get the current in the zeroth section by making $m = 0$ in (29) obtaining

$$A_0 = \frac{E}{\bar{z}} \qquad (31)$$

where $\bar{z}$ has the value given in (30); and on replacing x in (30) by its value from (17) we obtain

$$\bar{z} = z_0 - z_1/2 - z_2 - b\epsilon^k = z_0 + z_i \text{ (say)}, \qquad (32)$$

where

$$z_i = -b\epsilon^k - z_2 - z_1/2 \qquad (33)$$

The result contained in (33) may be otherwise written by use of (12) and (3) and becomes

$$z_i = \pm\sqrt{\frac{(z_1 + 2z_2)^2}{4} - b^2} \qquad (34)$$

(The sign is to be so chosen as to make the real part positive.)

It is to be noted that z_i as given by (33), *or* (34), *is the input impedance, or surge impedance, of the line of non-reflective output impedance, or of the infinite line, for by* (32) *and* (31) *z_i is the impedance by which the line exclusively of z_0 may be replaced without changing the current in the zeroth section.*

261. Complex Reflection Coefficient at the Remote End (Output End) of the Line.—We have seen in Art. **258** that the complex reflection coefficient at the remote end of the line is Y, defined in (22). Replacing y in (22) by its value from (18), we have

$$Y = -\frac{b\epsilon^{-k} + z_1/2 + z_2 - z_T}{b\epsilon^{k} + z_1/2 + z_2 - z_T} \tag{35}$$

Now by (10) and (3)

$$b\epsilon^{-k} = -b\epsilon^{k} - z = -b\epsilon^{k} - z_1 - 2z_2.$$

Introducing this result into (35), we obtain

$$Y = -\frac{-b\epsilon^{k} - z_2 - z_T - z_1/2}{b\epsilon^{k} + z_2 - z_T + z_1/2} \tag{36}$$

In the light of (33) this becomes

$$Y = \frac{z_i - z_T}{z_i + z_T} \tag{37}$$

Equation (36) *or the alternative equation* (37) *gives the complex reflection coefficient Y at the remote end of the line. In* (37) *z_i is the input impedance, or surge impedance, of an infinite line made up of the same kind of sections, and z_T is the complex impedance of the output terminal apparatus.*

262. Complex Reflection Coefficient at the Input End of the Line.—Since X differs from Y only in having z_0 in place of z_T we have by similarity with (36) and (37)

$$X = -\frac{-b\epsilon^{k} - z_2 - z_0 - z_1/2}{b\epsilon^{k} + z_2 - z_0 + z_1/2} \tag{38}$$

$$= \frac{z_i - z_0}{z_i + z_0} \tag{39}$$

Equation (38) *or* (39) *gives the complex reflection coefficient at the input end of the line. In* (39) *z_i is the complex input impedance of an infinite line made up of the same kind of sections, and z_0 is the complex impedance of the input terminal apparatus.*

263. Attenuation Constant per Section and Phase Lag per Section for the Current.—The constant k that enters in the various exponential quantities is called *the complex attenuation*

constant of the current per section of the line. The value of k is given in (11), from which it is seen that k is in general a complex quantity. Let us indicate its real and imaginary parts by writing

$$k = a + j\varphi \tag{40}$$

where a and φ are both real quantities and a is positive by note following (13).

We have seen above that each step from section to section of the line multiplies the complex amplitude by the factor ϵ^{-k}.

Now

$$\epsilon^{-k} = \epsilon^{-a}\epsilon^{-j\varphi} \tag{41}$$

whence it appears that

a = the *real* attenuation constant of the current introduced per section of the line, and

φ = retardation angle introduced per section into the phase of the current.

This is made evident as follows:

In the case of an infinite line or of a line whose reflection coefficient at the remote end is zero, by (29),

$$A_m = \frac{E}{\bar{\bar{z}}}\epsilon^{-km}.$$

$$= \frac{E}{\overline{\overline{R}+j\overline{X}}}\epsilon^{-am}\epsilon^{-j\phi m},$$

in which the complex quantities $\bar{z}$ and k have been replaced by

$$\bar{z} = \overline{R} + j\overline{X} \text{ (say)}$$

$$k = a + j\varphi \text{ as in (40)}$$

Now by Chapter IV, we may write

$$\bar{z} = \overline{R} + j\overline{X} = Z\epsilon^{j\Theta'}$$

where

$$\overline{Z} = \sqrt{\overline{R}^2 + \overline{X}^2} \text{ and } \tan\Theta' = \overline{X}/\overline{R}.$$

Substituting these values into the expression for A_m, we obtain

$$A_m = \frac{E}{\overline{\overline{Z}}}\epsilon^{-am}\epsilon^{-j(\Theta'+\phi m)},$$

whence by (1)

$$i_m = \frac{E}{\overline{\overline{Z}}}\epsilon^{-am}\epsilon^{j(\omega t-\Theta'-\phi m)}.$$

If now our original impressed e.m.f. e is

$$e = E \sin \omega t, \text{ instead of } e = E\epsilon^{j\omega t},$$

the expression for i_m would have the exponential with imaginary exponent replaced by a sine function, giving

$$i_m = \frac{E}{\overline{Z}} \epsilon^{-am} \sin \{\omega t - \Theta' - \phi m\} \tag{41a}$$

Equation (41*a*) *gives the real current in the mth section of a line without reflection at the output end, and shows that a is the attenuation constant and ϕ the retardation angle* (*of current*) *per section of the line.*

Determination of a and φ.—Let us now determine a and φ. In the expression for k, equation (11) there enters the complex quantity $-z/b$. Let us explicitly designate this complex quantity as

$$-z/2b = P + jU \tag{42}$$

where P and U are both real quantities, and where

$$P = \text{the real part of } -z/2b,$$

and

$$jU = \text{the imaginary part of } -z/2b.$$

Then by (11)

$$\cosh (a + j\varphi) = P + jU \tag{43}$$

Expanding the hyperbolic cosine into

$$\cosh a \cos \varphi + j \sinh a \sin \varphi,$$

and equating real and imaginary parts of (43) we have

$$\cosh a \cos \varphi = P \tag{44}$$

$$\sinh a \sin \varphi = U \tag{45}$$

These equations may be solved for a and φ by squaring both sides of the two equations and replacing

$$\cos^2\varphi \text{ by } 1 - \sin^2\varphi$$

and

$$\cosh^2 a \text{ by } 1 + \sinh^2 a,$$

giving

$$(1 + \sinh^2 a)(1 - \sin^2\varphi) = P^2$$

and

$$\sinh^2 a \sin^2\varphi = U^2.$$

Treating these last two equations as simultaneous, and elimination so as to solve for a and φ, we obtain

$$a = \sinh^{-1}\left\{+\sqrt{\pm\sqrt{U^2 + \frac{(1 - U^2 - P^2)^2}{4}} - \frac{1 - U^2 - P^2}{2}}\right\} \tag{46}$$

$$\varphi = \sin^{-1}\left\{+\sqrt{\pm\sqrt{U^2 + \frac{(1 - U^2 - P^2)^2}{4}} + \frac{1 - U^2 - P^2}{2}}\right\} \tag{47}$$

Let us now write as an abbreviation

$$V = \frac{1 - U^2 - P^2}{2} \tag{48}$$

then

$$a = \sinh^{-1}\{+\sqrt{\pm\sqrt{U^2 + V^2} - V}\} \tag{49}$$

$$\varphi = \sin^{-1}\{+\sqrt{\pm\sqrt{U^2 + V^2} + V}\} \tag{50}$$

Equations (46) *and* (47), *or the alternative equations* (49) *and* (50) *give a, which is the real attenuation constant for the current per section of the line, and φ, which is the angle of retardation introduced into the phase of the current per section of the line. The value of V is given in* (48). *The values of U and P are to be obtained from* (42).

In regard to the sign before the inner radical, it is to be noted that the same sign is to be used in the equation for a and the equation for φ, in order to satisfy (45), *and that this sign is to be chosen so as to make a and φ both real quantities.*

II. RESISTANCELESS LINES. FILTER ACTION

264. In a Resistanceless Line $-z/2b$ is Real.—In equations (3) we have used the notation

$$z = z_1 + 2z_2, \text{ and } b = Mj\omega - z_2,$$

where z_1 and z_2 are respectively the complex series impedances and the complex shunt impedances of the system, as illustrated in Fig. 1. If we suppose that the resistances are zero, these complex impedances may be replaced by j times the reactances, so that
if

$$R_1 = 0, \text{ and } R_2 = 0 \tag{51}$$

then

$$z = j(X_1 + 2X_2) \text{ and } b = j(M\omega - X_2) \tag{52}$$

and equation (42) becomes

$$P + jU = \frac{X_1 + 2X_2}{2(X_2 - M\omega)} = P_0 \text{ (say)} \tag{53}$$

where P_0 is seen to be real, and

$$U = 0 \tag{54}$$

Let us note now that (44) and (45) become, in this case,

$$\cosh a \cos \varphi = P_0 \tag{55}$$

$$\sinh a \sin \varphi = 0 \tag{56}$$

In case the series and shunt elements of the line are both resistanceless, the quantities a and φ satisfy (55) *and* (56), *and are easily determined in the following section.*

265. Determination of a and φ for a Resistanceless Line. The solution of (55) and (56) are seen to satisfy one or the other of the following conditions:

Either $\sin \varphi = 0$, then $\cos \varphi = \pm 1$, and $\cosh a = \pm P_0$ (57)

or

$$\sinh a = 0, \text{ then } \cosh a = 1, \text{ and } \cos \varphi = P_0 \tag{58}$$

We can distinguish between these two cases and can also determine the sign to use in (57) by noting that since a and φ are real

$$\cosh a \geqq 1 \tag{59}$$

$$[\cos \varphi] \leqq 1 \tag{60}$$

We shall need to distinguish three cases, according to the numerical value of P_0, as follows

Case I. $-1 \leqq P_0 \leqq +1,$

Case II. $+1 \leqq P_0,$

Case III. $P_0 \leqq -1.$

Equations (57) and (58) in view of (59) and (60) give for the three cases the following unique results.

Case I. If $-1 \leqq P_0 \leqq +1$, then

$$a = 0, \quad \text{and } \varphi = \cos^{-1} P_0 \tag{61}$$

Case II. If $+1 \leqq P_0$, then

$$\varphi = 0 \quad \text{and } a = \cosh^{-1}P_0 \tag{62}$$

Case III. If $P_0 \leqq -1$, then

$$\varphi = \pi \quad \text{and } a = \cosh^{-1}\{-P_0\} \tag{63}$$

Equations (61), (62), *and* (63) *give the values of a, which is the real attenuation constant per section, and of φ, which is the retardation introduced into the phase of the current per section of the line—in the case of a resistanceless line.*

266. Filter Action of the Resistanceless Line.—It is to be noted that the quantity P_0, as defined in (53) is determined by the reactances of the series elements and of the shunt elements and by the mutual inductance between adjacent series elements. These reactances and the mutual inductance term as it enters into (53) in general involve the angular velocity of the impressed e.m.f.; that is to say

$$P_0 = f(\omega),$$

where

$$\omega = \text{angular velocity of impressed e.m.f.}$$

For those values of ω that bring P_0 into the range of values of P_0 specified in Case I, currents are produced in the line that pass through it without attenuation, when the line is resistanceless, so that except for the effects of reflections the current in the nth section has the same amplitude as in the zeroth section.

On the other hand, for those values of ω that bring P_0 into the ranges specified by Case II and Case III, the attenuation constant a is not zero, so that with a sufficiently large number of sections, different for the different frequencies, any given frequency in these ranges not included in Case I will produce currents that are attenuated to any desired small fraction of the current in the zeroth element.

We shall now examine this filter action with respect to three special types of line.

267. Filter Action of Three Types of Resistanceless Line. The three special types to be investigated are shown in Figs. 2, 3, and 4.

In Fig. 2, which we shall call Type I, the line consists of capacities C_1 in series and inductances L_2 in shunt. The end sections in order to be sections of half impedance must be of capacity $2C_1$.

In Fig. 3 the line, which we shall classify as Type II, consists of inductance L_1 as series elements, and capacities C_2 as shunt elements, and terminates in inductances of $L_1/2$. Both Type I and Type II are examples of what is called a line of Π-sections

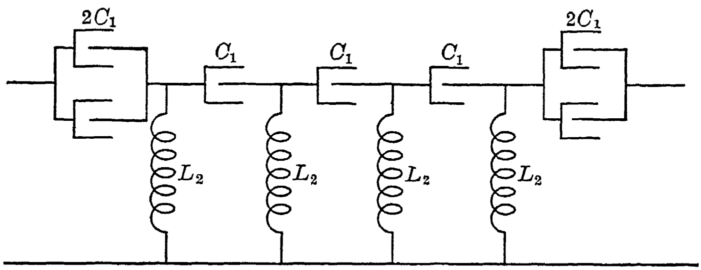

FIG. 2.—Line of Type I.

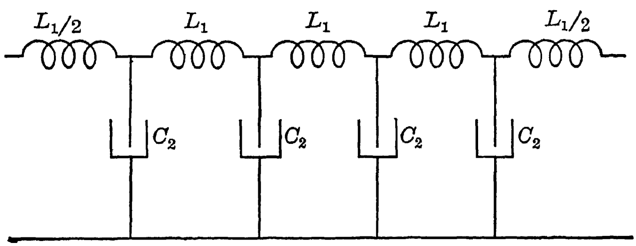

FIG. 3.—Line of Type II.

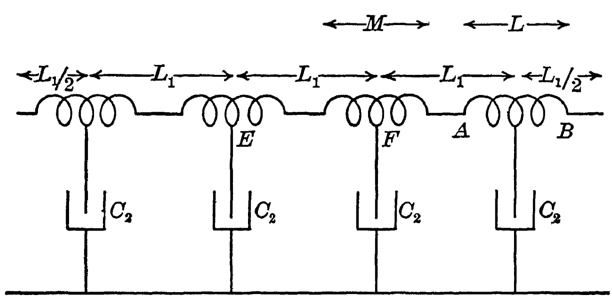

FIG. 4.—Line of Type III.

The line in Fig. 4, designated Type III, is similar to Type II except that there is mutual inductance M between the parts of coils common to two loops. The condensers C_2 are tapped to the mid points of these coils. It is to be noted that while the inductance per loop is L_2, this is not the inductance per coil.

The inductance per coil will be called L, in Type III. The line of Type III is called a line of T-sections.

268. Reference to Type I.—We shall now determine P_0 for Type I. In this case the reactances are

$$X_1 = -1/C_1\omega,\ X_2 = L_2\omega,\ M = 0 \tag{64}$$

These values substituted into (53) give

$$P_0 = 1 - \frac{1}{2L_2C_1\omega^2} \tag{65}$$

Then by (61),

$$a = 0,$$

provided

$$-1 \leqq 1 - \frac{1}{2L_2C_1\omega^2} \leqq +1;$$

that is provided

$$\frac{1}{2\sqrt{L_2C_1}} \leqq \omega \leqq \infty \tag{66}$$

If as an abbreviation we write

$$\frac{1}{L_2C_1} = \Omega^2 \tag{67}$$

the inequality (66) becomes

$$\frac{1}{2} \leqq \frac{\omega}{\Omega} \leqq \infty \tag{68}$$

With a system of Type I, Fig. 2, which is supposed to have zero resistance, the attenuation is zero for all currents of angular velocity greater than $\Omega/2$, where Ω has the definition given in (67). On the other hand, for all currents of angular velocity ω less than $\Omega/2$, it is seen by reference to (63) that the attenuation constant is

$$\begin{aligned} a &= \cosh^{-1}\left\{\frac{1}{2L_2C_1\omega^2} - 1\right\} \qquad (69) \\ &= \cosh^{-1}\left\{\frac{\Omega^2}{2\omega^2} - 1\right\} \end{aligned}$$

This type of circuit lets through without attenuation frequencies higher than a specified value, and attenuates frequencies lower than the specified value, and attenuates them more the lower their frequencies.

Computations and curves will be given later.

269. Reference to Type II.—In case of the resistanceless system of Type II, Fig. 3, the impedances are

$$X_1 = L_1\omega, \quad X_2 = -1/C_2\omega, \quad M = 0 \tag{70}$$

so that

$$P_0 = 1 - \frac{L_1C_2\omega^2}{2} \tag{71}$$

In this type of system, we shall have, by (61),

$$a = 0 \tag{72}$$

provided

$$-1 \leqq 1 - \frac{L_1C_2\omega^2}{2} \leqq +1 \tag{73}$$

that is, provided

$$2 \geqq \frac{\omega}{\Omega} \geqq 0 \tag{74}$$

where now

$$\Omega^2 = 1/L_1C_2 \tag{75}$$

On the other hand, if

$$\frac{\omega}{\Omega} > 2, \text{ then } a = \cosh^{-1}\left\{\frac{L_1C_2\omega^2}{2} - 1\right\}$$

$$= \cosh^{-1}\left\{\frac{\omega^2}{2\Omega^2} - 1\right\}. \tag{76}$$

With a resistanceless system of Type II, Fig. 3, the attenuation is zero for currents of all angular velocities ω *less than* 2Ω, *where* Ω *has the value given in* (75). *On the other hand, for currents of angular velocities greater than* 2Ω *the attenuation is given by* (76) *and increases with increasing value of the angular velocity.*

This type of circuit lets through frequencies lower than the specified value without attenuation, and attenuated currents of frequencies higher than the specified value.

270. Reference to Type III.—If a line of Type III, Fig. 4, is made up of elements of zero resistance, the reactances are

$$X_1 = L_1\omega, \quad X_2 = -1/C_2\omega, \quad M = M \tag{77}$$

If we think of the inductance elements as made up of coils, as AB, having inductance L and tapped at their mid points for the attachment of the condensers, it is to be noted that the inductance L_1 is made up of two coils in series, each of which has the inductance of a half-coil AB. The mutual inductance M is

the mutual inductance between two half coils; whence, if there is no magnetic leakage,

$$M = L_1/2 \tag{78}$$

If there is magnetic leakage,

$$M < L_1/2 \tag{79}$$

Let us now introduce the coefficient of coupling τ, between two adjacent loops.

Then

$$\tau^2 = M^2/\ L_1L_1,$$

whence

$$\tau = M/L_1 \leqq 1/2 \tag{80}$$

Now introducing (77) into (53) we obtain

$$P_0 = \frac{L_1\omega - \dfrac{2}{C_2\omega}}{2\left(-\dfrac{1}{C_2\omega} - M\omega\right)} = \frac{L_1C_2\omega^2 - 2}{-2(1 + MC_2\omega^2)} \tag{81}$$

Introducing this value of P_0 into (61), we obtain

$$a = 0,$$

provided

$$-1 \leqq \frac{L_1C_2\omega^2 - 2}{-2 - 2MC_2\omega^2} \leqq +1 \tag{82}$$

that is, provided

$$\frac{2}{\sqrt{(L_1 - 2M)C_2}} \geqq \omega \geqq 0 \tag{83}$$

In terms of τ, the inequality (83) can be written

$$\frac{2}{\sqrt{L_1C_2}\sqrt{1 - 2\tau}} \geqq \omega \geqq 0 \tag{84}$$

Corollary.—If the coils, as AB, have no magnetic leakage, then by (78), equation (84) becomes

$$\infty \geqq \omega \geqq 0 \tag{85}$$

With a resistanceless line of Type III, Fig. 4, having mutual inductance between adjacent loops, currents are transmitted unattenuated for all angular velocities given by (83) *or* (84). *If the coils*

have zero magnetic leakage, then by (85) all possible frequencies are transmitted without attenuation. Such a line is not a good filter, but will be found useful for its retardation properties when it is desired to transmit with suitable retardation all frequencies.

III. RESISTANCELESS LINES. TERMINAL IMPEDANCE

271. Surge Impedance of the Three Types of Resistanceless Lines.—In order to adapt a line to its terminal conditions, or to adapt the terminal conditions to the line, it is important to choose the constants so that the line will transmit as large a current as possible with the frequencies that it is desired to transmit. This means that reflection at the output terminal apparatus should be avoided and that the equivalent impedance of the whole line, with its non-reflective output apparatus should be adapted to the impedance of the input terminal apparatus.

This requires the determination of the quantity that we have called z_i, where

z_i = the surge impedance = the impedance by which a line of an infinite number of sections, or of a finite number of sections with a non-reflective output impedance, can be replaced without changing the current in the zeroth section.

We have found a general expression for z_i in equation (34), which is

$$z_i = \sqrt{\frac{(z_1 + 2z_2)^2 - 4b^2}{4}} \tag{86}$$

We shall now determine z_i for the three types of resistanceless line given in Figs. 2, 3, and 4. In all of these types, since the resistances are zero, we may write

$$z_1 = jX_1,\ z_2 = jX_2, \text{ and } b = j(M\omega - X_2) \tag{87}$$

Introducing these values into (86), we obtain

$$z_i = \frac{1}{2}\sqrt{-X_1^2 - 4X_1X_2 + 4M^2\omega^2 - 8M\omega X_2} \tag{87a}$$

for

$$R_1 = 0 = R_2$$

Now introducing the values of X_1, X_2, and M for Types I, II, and III respectively, as given in (64), (70), and (77), we obtain

For Type I,

$$z_i = \sqrt{\frac{L_2}{C_1} - \frac{1}{4C_1^2\omega^2}} \tag{88}$$

For Type II,

$$z_i = \sqrt{\frac{L_1}{C_2} - \frac{L_1^2\omega^2}{4}} \tag{89}$$

For Type III,

$$z_i = \sqrt{\frac{L_1 + 2M}{C_2} + \frac{(4M^2 - L_1^2)\omega^2}{4}} \tag{90}$$

For Type III, with no magnetic leakage, $2M = L_1$, and

$$z_i = \sqrt{2L_1/C_2} = \sqrt{L/C_2} \tag{91}$$

where

L = inductance per coil of Type III, Fig 4.

Equations (88), (89), *and* (90) *give the surge impedance, or equivalent impedance of a line with non-reflective output terminal apparatus, for Type I, Type II, and Type III, respectively.*

Equation (91) *gives the corresponding quantity for Type III if the coils have no magnetic leakage. In this case it is seen that z_i is of the character of a pure resistance and is independent of the frequency.*

In (88), (89), *and* (90) *z_i is also a real quantity, and is of the character of a pure resistance, but in general this equivalent resistance involves the angular velocity and is different for currents of different frequencies. It will be shown later that by choosing the inductances and capacities small, while keeping their ratios large the terms involving angular velocity can be made negligible over considerable ranges of frequencies.*

272. Condition for Non-reflective Output.—In (37), we have seen that the complex reflection coefficient Y at the output terminal of the line is

$$Y = \frac{z_i - z_T}{z_i + z_T}.$$

This is zero, if

$$z_T = z_i \tag{92}$$

Equation (92) *shows that for no reflection at the junction of the line with the output apparatus the complex impedance of the output apparatus z_T must be equal to the surge impedance z_i of the line. This is true whether the line has resistance or not.*

To adapt this result to the three special types of line used in the illustration, it is only necessary to replace z_i in (92) by its known values for the three types.

273. Condition for Non-reflection at the Input Terminal Apparatus.—Likewise, by (39), whether the line is resistanceless or not, we can make the complex reflection coefficient X at the input end zero, if we can make

$$z_0 = z_i \tag{93}$$

To make the line non-reflective at the input terminal apparatus it is necessary to make z_0, which is the impedance of the input apparatus, equal to the surge impedance z_i of the line. This is true whether the line is resistanceless or not.

IV. RESISTANCELESS LINES. RETARDATION COMPUTATIONS

274. Retardation per Section of Resistanceless Line.—In equations (61), (62) and (63) we have found that if

$$-1 \leqq P_0 \leqq +1, \quad a = 0, \quad \text{and} \quad \varphi = \cos^{-1} P_0 \tag{94}$$

if

$$P_0 \geqq +1, \quad a = \cosh^{-1} P_0, \quad \varphi = 0 \tag{95}$$

if

$$P_0 \leqq -1, \quad a = \cosh^{-1}\{-P_0\}, \quad \varphi = \pi \tag{96}$$

In these equations φ is the angle of lag introduced into the current by each section of the line.

It is seen that with the resistanceless line, for the range of frequencies within which P_0 satisfies the inequality in (94), the angle of lag per section is equal to the anticosine of P_0, and is in general a complicated function of the frequency, since P_0 is a function of ω.

Outside of this range of frequencies the angle of lag per section is a constant 0, under condition (95), or a constant π, under condition (96).

Before discussing further the lag angle φ, let us introduce tables giving a and φ for the three types of lines shown in Figs. 2, 3, and 4.

In compiling Tables I, II, and III, arbitrary values were taken for the quantities in the first column. Corresponding to these arbitrary quantities, the quantities in the other columns were computed by equations (94) to (96). In the second columns

of Tables I and II is given the attenuation constant a for the current, per section of the line. This quantity for Table III is not given, since it is zero throughout.

Table I.—Resistanceless Line of Type I

Attenuation Constant a and Retardation Angle φ of Current per Section for Different Angular Velocities ω of the Current

L_2 = Shunt Inductance per Section
C_1 = Capacity in Series per Section

$\omega\sqrt{L_2C_1}$	a	φ radians	ϵ^{-10a}
0 0	∞	-3 1416 $= \pi$	0 00000
0 2	3.13	-3 1416 $= \pi$	0 00000+
0 3	2 20	-3 1416 $= \pi$	0 00000+
0 4	1.385	-3 1416 $= \pi$	0 00000+
0 45	0 93	-3 1416 $= \pi$	0 00009
0 46	0 830	-3 1416 $= \pi$	0 0002
0 47	0 710	-3 1416 $= \pi$	0 0008
0 48	0 575	-3 1416 $= \pi$	0 0032
0 495	0 288	-3 1416 $= \pi$	0 056
0 497	0 220	-3 1416 $= \pi$	0 111
0 499	0 130	-3 1416 $= \pi$	0 273
0 500	0 000	-3 1416 $= \pi$	1 00
0 501	0 000	-3 0524	1 00
0 503	0 000	-2 9234	1 00
0 505	0 000	-2.8606	1 00
0 6	0 000	-1 971	1 00
0 7	0 000	-1 591	1 00
0 8	0 000	-1 350	1 00
0 9	0 000	-1 178	1 00
1 0	0 000	-1 047	1 00
2.0	0 000	-0 505	1 00
3 0	0 000	-0 335	1.00
4 0	0.000	-0 250	1 00
∞	0 000	-0 000	1 00

In the last columns of Tables I and II there is compiled the quantity ϵ^{-10a}, which is obtained on the supposition of line of ten sections or more. This quantity ϵ^{-10a} is the ratio of the current amplitude in the tenth section to the current amplitude in the zeroth section, and shows the sharpness with which the line of ten sections cuts off frequencies near the limit of frequencies for which the attenuation is zero. As an example, if we take Table I, currents of all angular velocities from ω

equal infinity to ω equal $0.5/\sqrt{L_2C_1}$ are transmitted unattenuated; while, on the other hand, if ω is equal to $0.497/\sqrt{L_2C_1}$ the current in the tenth section is only 11 per cent. of the current in the zeroth section, and if ω is $0.48/\sqrt{L_2C_1}$ the current in the tenth section is only 3/10 of one per cent. of the current in the zeroth section. In a similar manner, one may interpret the values in the last column of Table II.

Table II.—Resistanceless Line of Type II

Attenuation Constant a, Retardation Angle φ in Radians, Time Lag T in Seconds—Each for One Section, and Having Reference to Current

L_1 = Series Induction per Section
C_2 = Shunt Capacity per Section

$\omega\sqrt{L_1C_2}$	a	φ radians	$T/\sqrt{L_1C_2}$	ϵ^{-10a}
0 00	0 00	0 000	1 0000	1 000
0 05	0 00	0 05000	1 0000	1.000
0 10	0 00	0 10007	1 0007	1 000
0 15	0 00	0 15025	1 0017	1 000
0 20	0 00	0 2004	1 002	1.000
0 25	0 00	0 2507	1 003	1 000
0 30	0 00	0 3011	1 004	1 000
0 35	0 00	0 3517	1 005	1 000
0 40	0 00	0 4029	1.007	1 000
0 50	0 00	0 5053	1 011	1 000
0 60	0 00	0 6095	1 016	1 000
0.70	0 00	0 7151	1.022	1 000
0 80	0 00	0 826	1 032	1 000
1 00	0 00	1.05	1 050	1 000
1.20	0 00	1 29	1.075	1 000
1.40	0 00	1.55	1 109	1 000
1.60	0 00	1.85	1 156	1 000
1 80	0 00	2 23	1 239	1 000
2 00	0 00	3 1416 $=\pi$	1 570	1 000
2 001	0.06	3 1416 $=\pi$	......	0 549
2 002	0.09	3 1416 $=\pi$	......	0 407
2 003	0 11	3 1416 $=\pi$	......	0 333
2.004	0 13	3 1416 $=\pi$	......	0 273
2 005	0 15	3 1416 $=\pi$	......	0 223
2 01	0.20	3 1416 $=\pi$	......	0 135
2 02	0 29	3.1416 $=\pi$	......	0 055
2 04	0.40	3.1416 $=\pi$	......	0 0183
2 10	0 63	3 1416 $=\pi$	......	0.00184
2 20	0 89	3 1416 $=\pi$	......	0 00014

Table III.—Resistanceless Line of Type III, with no Magnetic Leakage Retardation Angle φ of Current per Section, and Time Lag T of Current per Section. The Attenuation Constant is Zero Throughout

L_1 = Series Inductance per Loop
L = Series Inductance per Coils
C_2 = Shunt Capacity per Coil

$\omega\sqrt{L_1C_2}$	$\omega\sqrt{LC_2}$	φ radians	$T/\sqrt{LC_2}$
0 0000	0 0000	0 0000	1 0000
0 05	0 07071	0 07071	1 0000
0 10	0 14142	0 1411	0 998
0 15	0 21213	0 2117	0 998
0 20	0 28284	0 2822	0.9978
0 25	0 35361	0.3503	0 9906
0 30	0 4243	0 4180	0 985
0 35	0 4950	0 4851	0 980
0 40	0 5657	0 5512	0 975
0 60	0 8485	0 8029	0 947
0 80	1 1314	1 0297	0 910
1 00	1 4142	1.2305	0 870
1 20	1 6970	1 4070	0 829
1 40	1 9799	1 5600	0 788
1 50	2 1313	1 6290	0 768
1 60	2 2627	1 7046	0 753
1 80	2 5456	1.810	0.710

275. Line of Constant Time Lag per Section over Significant Range of Frequencies.—The tables also contain values of the current-lag-angle per section of the line in the columns headed φ. A related quantity is the quantity T, which is the number of seconds by which the current in any section lags behind the current in the preceding section. The quantity T is related to φ by the equation

$$\varphi = \omega T \tag{97}$$

obtained as follows: If the angle of lag is φ radians, and the angular velocity of the current is ω radians per second, the time T is such that the system would describe an angle φ in time T at angular velocity ω, provided $\varphi = \omega T$. We should then be able to tabulate T by dividing φ by ω, but since ω is given only as a factor in the quantity at the heading of the second column, we have divided the numbers in this column into the corresponding numbers in the column headed φ, obtaining

$$\frac{\varphi}{\omega\sqrt{L\,C_2}} = \frac{T}{\sqrt{L\,C_2}} \tag{98}$$

An important result obtained in this way is that *for Line of Type II, and for small values of* $\omega/\sqrt{L_1C_2}$ *the time lag T per section is approximately constant and equal to* $\sqrt{L_1C_2}$.

For a line of Type III (assumed to have zero magnetic leakage in the coils, and for small values of $\omega\sqrt{LC_2}$, *the time lag T per section is approximately constant and equal to* $\sqrt{LC_2}$. *In this latter result L is the inductance per coil and not the inductance per loop of Type III, which is the system containing the mutual inductance between the loops. The inductance per loop is* $L_1 = L/2$.

We have thus obtained a method, using systems of Type II or of Type III, of obtaining per section of line a time lag of current substantially independent of frequency over a significant range of frequencies.

Apart from the use of the tables, we can prove this result theoretically, by expanding in series the anticosine of P_0 in equation (94), and neglecting certain higher powers of small quantities. We shall perform this operation, later, in connection with lines of sensible resistance, to be studied in the next few pages.

V. LINES WITH RESISTANCE. TYPE I

276. General Equations. Types I and II.—To determine a and φ, when the line has resistance, we must return to equations (49) and (50), which are general. In order to get U and V for use in these equations, we need to start with (42), which in view of (3) becomes

$$\frac{-z}{2b} = \frac{z_1 + 2z_2}{2(z_2 - Mj\omega)} = P + jU \qquad (99)$$

which may be regarded as defining the real quantities P and U.

In lines of Types I and II, M is zero, so that for these cases

$$1 + \frac{z_1}{2z_2} = P + jU \qquad \text{(Type I, or Type II)} \qquad (100)$$

277. Determination of a and ϕ for Line of Type I.—For Type I, if we take account of the resistance in the inductance coils, we have

$$z_1 = -j/C_1\omega, \quad z_2 = R_2 + jL_2\omega \qquad (101)$$

These values inserted into (100), give after rationalizing, and equating real and imaginary parts,

$$P = 1 - \frac{L_2\omega}{2C_1\omega(R_2^2 + L_2^2\omega^2)} \tag{102}$$

$$U = \frac{-R_2}{2C_1\omega(R_2^2 + L_2^2\omega^2)} \tag{103}$$

Then by the use of (48), we obtain

$$V = \frac{4L_2C_1\omega^2 - 1}{8C_1^2\omega^2(R_2^2 + L_2^2\omega^2)} \tag{104}$$

These values of U and V can be put into a form in which the relative size of terms is more evident, by introducing the abbreviations

$$\eta_2 = R_2/L_2\omega, \quad \text{and } \theta = L_2C_1\omega^2 \tag{105}$$

then

$$U = \frac{-\eta_2}{2\theta(1 + \eta_2^2)} \tag{106}$$

$$V = \frac{4\theta - 1}{8\theta^2(1 + \eta_2^2)} \tag{107}$$

Introducing these values into (49) and (50), we have

$$a = \sinh^{-1} \left\{\frac{1}{2\theta\sqrt{2(1 + \eta_2^2)}}\sqrt{\pm\sqrt{16\theta^2\eta_2^2 + (1 - 4\theta)^2} + (1 - 4\theta)}\right\} \tag{108}$$

$$\phi = \sin^{-1} \left\{\frac{1}{2\theta\sqrt{2(1 + \eta_2^2)}}\sqrt{\pm\sqrt{16\theta^2\eta_2^2 + (1 - 4\theta)^2} - (1 - 4\theta)}\right\} \tag{109}$$

Equations (108) *and* (109) *give the values of the attenuation constant a and the retardation angle φ per section of the line of Type I. In these equations η_2 and θ have the values given in* (105). *Regarding the sign before the inner radical, it is to be noted that the same sign is to be used in both equations, and this sign is to be selected so as to make both a and φ real quantities.*

278. Determination of Surge Impedance for Line of Type I with Resistance.—The general expression for surge impedance is given in (34), which in view of (101) and (3) gives

$$z_i = \sqrt{\frac{\{-j/C_1\omega + 2(R_2 + jL_2\omega)\}^2}{4} - \{R_2 + jL_2\omega\}^2}$$

$$= \frac{1}{2}\sqrt{-\frac{1}{C_1^2\omega^2} - \frac{4jR_2}{C_1\omega} + \frac{4L_2}{C_1}}$$

whence

$$z_i = \sqrt{\frac{L_2}{C_1}}\sqrt{1 - \frac{1}{4L_2C_1\omega^2} - \frac{jR_2}{L_2\omega}} \tag{110}$$

Equation (110) gives the general expression for the surge impedance of a line of Type I containing resistance in the inductance coils.

279. Approximate Treatment for Small Values of η_2 and for 4θ not too Near Unity.—We may obtain simplified expressions for (108) and (109) for small values of η_2^2 and for 4θ greater than and not too near unity by expanding the radicals in these equations and neglecting higher terms. Assuming, to begin, that

$$16\theta^2\eta_2^2 < (1 - 4\theta)^2,$$

and expanding the radicals, we obtain

$$a = \sinh^{-1}\left\{\frac{1}{2\theta\sqrt{2(1+\eta_2^2)}}\left[\pm(1-4\theta)\left(1 + \frac{8\theta^2\eta_2^2}{(1-4\theta)^2} - \frac{32\theta^4\eta_2^4}{(1-4\theta)^4} + \ldots\right) + (1-4\theta)\right]^{1/2}\right\} \tag{111}$$

and

$$\phi = \sin^{-1}\left\{\frac{1}{2\theta\sqrt{2(1+\eta_2^2)}}\left[\pm(1-4\theta)\left(1 + \frac{8\theta^2\eta_2^2}{(1-4\theta)^2} - \frac{32\theta^4\eta_2^4}{(1-4\theta)^4} + \ldots\right) - (1-4\theta)\right]^{1/2}\right\} \tag{112}$$

In order to make a and φ real we must use the negative sign in the bracket, whenever $1 - 4\theta$ is negative; and must use the positive sign, whenever $1 - 4\theta$ is positive.

Let us now assume that

$$\eta_2^2/2 << 1, \text{ and } \frac{4\theta^2\eta_2^2}{(1-4\theta)^2} << 1 \tag{113}$$

Under these conditions we have by (111) and (112)

$$a = \sinh^{-1}\left\{\frac{\eta_2}{\sqrt{4\theta - 1}}\right\}, \text{ and } \varphi = \sin^{-1}\left\{\frac{\sqrt{4\theta-1}}{2\theta}\right\}, \text{ for } 4\theta > 1 \tag{114}$$

and

$$a = \sinh^{-1}\left\{\frac{\sqrt{1-4\theta}}{2\theta}\right\}, \text{ and } \varphi = \sin^{-1}\left\{\frac{\eta_2}{\sqrt{1-4\theta}}\right\}, \text{ for } 4\theta < 1 \tag{115}$$

In case the ratio of the resistance of the coils to their inductive reactance is small and in case 4θ *is not too near unity, so that the conditions* (113) *are fulfilled, equation* (114) *gives a and* φ *for* θ *greater than* 1/4, *and equation* (115) *gives the corresponding values of a and* φ *for* θ *less than* 1/4. *These results are for line of Type I.*

In regard to the attenuation constant it should be noticed from the formulas (114) and (115) that the former gives the case of low attenuation and the latter gives the case of high attenuation. The transition point is somewhere near $4\theta = 1$, but neither of these formulas can be used in this region because (113) fails there. We must go back to (108) and (109) if 4θ is nearly equal to unity. It is seen by (108) that the last term under the outer radical changes from a numerically subtractive term to a numerically additive term when 4θ passes from values greater than unity to values less than unity, and as 4θ goes on decreasing, a increases rapidly. We shall call the value of the frequency at which $4\theta = 1$ *the cut-off frequency.*

It will be understood that this cut-off frequency is not a point of discontinuity giving a sudden change of the attenuation with change of frequency. The increase of attenuation as we pass the cut-off frequency and pass into the region of frequencies that are more attenuated is rapid for low-resistance coils, and after a change of a few per cent. in frequency the attenuation for a line of five or ten sections may be such as to reduce the current to less than one per cent. of its value at the cut-off frequency. We shall later show this by numerical computations.

To complete the approximate treatment of this type of line (*Type I*) let us note that under conditions (113), equation (110) for the surge impedance becomes

$$z_i = \sqrt{\frac{L_2}{C_1}}\sqrt{1 - \frac{1}{4L_2C_1\omega^2}} \tag{116}$$

Introducing into this equation, the value of θ given in (105), we obtain

$$z_i = \sqrt{\frac{L_2}{C_1}}\sqrt{1 - \frac{1}{4\theta}} \tag{117}$$

Under condition (113) *the surge impedance of a line of Type I is given by* (117), *or by the alternative equation* (116). *This surge impedance is real and of the nature of a pure resistance, provided* 4θ *is greater than unity, and is imaginary, and therefore of the*

nature of a reactance if 4θ is less than unity. These equations are not to be used for 4θ too near to unity for then (113) *is not fulfilled. It is seen by* (116) *that the surge impedance is in general a function of the angular velocity ω.*

If, however,

$$\frac{1}{8\theta} << 1; \text{ that is, if } \frac{1}{8L_2C_1\omega^2} << 1 \tag{118}$$

equation (117) becomes

$$z_\iota = \sqrt{L_2/C_1} \tag{119}$$

which is independent of the frequency.

With a line of Type I, in case the conditions (118) *and* (113) *are fulfilled, equation* (119) *gives the surge impedance z_ι of the line. This is in the nature of a pure resistance independent of the frequency of transmitted current.*

We should here note also, for future use that by (93) the condition for non-reflection at the input apparatus is

$$z_0 = z_\iota,$$

and the condition for non-reflection at the output apparatus is

$$z_T = z_\iota,$$

where z_0 and z_T are the impedances of the input apparatus and the output apparatus respectively.

VI. LINES WITH RESISTANCE. TYPE II

280. Specific Values for Type II.—For a Line of Type II, as given in Fig. 3, the series and shunt complex impedances have the values

$$z_1 = R_1 + jL_1\omega, \qquad z_2 = -j/C_2\omega \tag{120}$$

where now the coils in the series impedances are supposed to have resistance R_1.

Introducing (120) into (100) we obtain

$$P + jU = 1 - \frac{L_1C_2\omega^2}{2} + \frac{jR_1C_2\omega}{2}.$$

If, now as abbreviations, we write

$$\eta_1 = R_1/L_1\omega, \qquad \Psi = L_1C_2\omega^2 \tag{121}$$

substitute these values into the preceding equation, and separate the result into real and imaginary parts, we obtain

$$P = 1 - \Psi/2$$
$$U = \eta_1\Psi/2 \tag{122}$$

and by the definition of V given in (48)

$$V = \frac{\Psi}{2}\left\{1 - \frac{\Psi}{4}(1 + \eta_1^2)\right\} \tag{123}$$

These values of U and V introduced into (49) and (50) give

$$a = \sinh^{-1}\left\{\sqrt{\frac{\Psi}{2}} \sqrt{\pm\sqrt{\eta_1^2 + \left\{1 - \frac{\Psi}{4}(1 + \eta_1^2)\right\}^2} - \left\{1 - \frac{\Psi}{4}(1 + \eta_1^2)\right\}}\right\} \tag{124}$$

$$\varphi = \sin^{-1}\left\{\sqrt{\frac{\Psi}{2}} \sqrt{\pm\sqrt{\eta_1^2 + \left\{1 - \frac{\Psi}{4}(1 + \eta_1^2)\right\}^2} + \left\{1 - \frac{\Psi}{4}(1 + \eta_1^2)\right\}}\right\} \tag{125}$$

Equations (124) *and* (125) *give the values of attenuation constant a and retardation angle φ per section of the line for a Line of Type II. The abbreviations employed are given in* (12). *The same sign must be employed before the inner radical in both equations, and that sign must be chosen to make a and φ both real quantities.*

281.—Determination of Surge Impedance for Line of Type II with Resistance.—When $M = 0$, equation (3) gives

$$b = -z_2.$$

This inserted into (34) gives the surge impedance

$$z_s = \tfrac{1}{2}\sqrt{z_1^2 + 4z_1z_2}$$

Introducing the values of z_1 and z_2 gives in (120), we obtain

$$z_s = \tfrac{1}{2}\sqrt{R_1^2 + 2jR_1L_1\omega - L_1^2\omega^2 - \frac{4jR_1}{C_2\omega} + \frac{4L_1}{C_2}};$$

whence

$$z_s = \sqrt{\frac{L_1}{C_2}}\sqrt{1 + \frac{R_1^2C_2\omega}{4L_1\omega} - \frac{L_1C_2\omega^2}{4} + j\left\{\frac{R_1C_2\omega}{2} - \frac{R_1}{L_1\omega}\right\}} \tag{125a}$$

Equation (125a) gives the surge impedance of a line of Type II with resistance in the inductance coils.

282. Approximate Treatment for Small Values of η_1, and for Ψ Less Than and not too Near to the Value 4.—If as a temporary abbreviation we put

$$A = 1 - \frac{\Psi}{4}(1 + \eta_1^2) \tag{126}$$

and assume

$$\eta_1^2 < A^2,$$

we may expand (124) and (125) into

$$a = \sinh^{-1}\left\{\sqrt{\frac{\Psi}{2}}\left[\pm A\left(1 + \frac{\eta_1^2}{2A^2} - \frac{\eta_1^4}{8A^4} + \ . \ . \ .\right) - A\right]^{\frac{1}{2}}\right\} \tag{127}$$

and

$$\varphi = \sin^{-1}\left\{\sqrt{\frac{\Psi}{2}}\left[\pm A\left(1 + \frac{\eta_1^2}{2A^2} - \frac{\eta_1^4}{8A^4} + \ . \ . \ \right) + A\right]^{\frac{1}{2}}\right\} \tag{128}$$

If A is positive, we use the positive sign, and if A is negative we use the negative sign, giving, in the case of positive A,

$$a = \sinh^{-1}\left\{\eta_1\sqrt{\frac{\Psi}{4A}\left(1 - \frac{\eta_1^2}{4A^2} + \ . \ . \ .\right)}\right\}, A > 0 \tag{129}$$

and

$$\varphi = \sin^{-1}\left\{\sqrt{\Psi A\left(1 + \frac{\eta_1^2}{4A^2} - \ . \ . \ .\right)}\right\}, A > 0 \tag{130}$$

Equations (129) and (130) reduce to

$$a = \frac{\eta_1\sqrt{\Psi}}{2} = \frac{R_1}{2}\sqrt{\frac{C_2}{L_1}}, \text{ and } \varphi = \omega\sqrt{L_1C_2} \tag{131}$$

provided

$$\frac{R_1^2}{8L^2\omega^2} << 1, \text{ and } \frac{L_1C_2\omega^2}{6} << 1 \tag{132}$$

In the case of small decrement and small value of $L_1C_2\omega^2$, as stipulated in (132), approximate values for a and φ for a line of Type II are given in (131). When the conditions (132) are not fulfilled, the exact equations (124) and (125) are to be used.

As in the case of line of Type I, other approximations to suit other conditions will be apparent to the reader.

It is to be noted also that under conditions (132), the equation (125a) for the surge impedance becomes

$$z_s = \sqrt{\frac{L_1}{C_2}} \tag{133}$$

With a line of Type II, in case conditions (132) *are satisfied, the surge impedance of the line is given approximately by* (133), *and is in the nature of a pure resistance independent of the frequency, so long as* ω *satisfies* (132).

VII. LINES WITH RESISTANCE. TYPE III

283. Determination of a and φ for Type III with Resistance. This type of line is shown diagrammatically in Fig. 4. If the resistance per loop is R_1, we have

$$z_1 = R_1 + jL_1\omega, \quad z_2 = -j/C_2\omega, \quad b = jM\omega + j/C_2\omega \tag{134}$$

By (42), in view of (3)

$$P + jU = -z/2b = 1 + \frac{z_1 + 2jM\omega}{2(z_2 - jM\omega)},$$

which by (134)

$$= 1 - \frac{(L_1 + 2M)\omega}{2\left(\frac{1}{C_2\omega} + M\omega\right)} + \frac{jR_1}{2\left(\frac{1}{C_2\omega} + M\omega\right)}.$$

We can make a simplification in this equation by introducing the inductance L of each of the whole coils, that are tapped at the middle. Since L_1 is twice the inductance of a half coil, and R_1 is twice the resistance of a half coil,

$$L = L_1 + 2M, \quad \text{and } R = R_1 \tag{135}$$

where L and R are now the inductance and resistance per coil of the system. Using these values, and equating real and imaginary parts of the equation preceding (135) we have

$$P = 1 - \frac{LC_2\omega^2}{2(1 + MC_2\omega^2)}, \quad U = \frac{RC_2\omega}{2(1 + MC_2\omega^2)} \tag{136}$$

We may now find V, as defined in (48), which is

$$V = \frac{1}{2}\frac{LC_2\omega^2}{1 + MC_2\omega^2}\left\{1 - \frac{LC_2\omega^2}{4(1 + MC_2\omega^2)}\left(1 + \frac{R^2}{L^2\omega^2}\right)\right\} \tag{137}$$

Let us now introduce as abbreviations

$$\eta = R/L\omega, \text{ and } Q = \frac{LC_2\omega^2}{1 + MC_2\omega^2} \tag{138}$$

then

$$U = \eta Q/2, \; V = \frac{Q}{2}\left\{1 - \frac{Q}{4}\left(1 + \eta^2\right)\right\} \tag{139}$$

By comparing these values of U and V with the corresponding quantities for Type II, as given in equations (122) and (123), we see that the values of U and V are analogues for the two cases. By replacing Ψ and η_1 in (124) and (125) by Q and η respectively, we obtain for the present case

$$a = \sinh^{-1}\left\{\sqrt{\frac{Q}{2}}\sqrt{\pm\sqrt{\eta^2 + \left\{1 - \frac{Q}{4}\left(1 + \eta^2\right)\right\}^2} - \left\{1 - \frac{Q}{4}\left(1 + \eta^2\right)\right\}}\right\} \tag{140}$$

$$\varphi = \sin^{-1}\left\{\sqrt{\frac{Q}{2}}\sqrt{\pm\sqrt{\eta^2 + \left\{1 - \frac{Q}{4}\left(1 + \eta^2\right)\right\}^2} + \left\{1 - \frac{Q}{4}\left(1 + \eta^2\right)\right\}}\right\} \tag{141}$$

Equations (140) *and* (141) *give the values of the attenuation constant a and the retardation angle φ per section of the line for a line of Type III. The abbreviations employed are given in* (138). *The same sign must be employed before the inner radical in both equations, and that sign must be selected to make a and φ both real quantities.*

284. Surge Impedance for a Line of Type III with Resistance. The substitution of (134) and (135) into (34) gives for the surge impedance

$$z_1 = \frac{\sqrt{\dfrac{L}{C_2}}}{\sqrt{1 + \dfrac{R^2C_2\omega}{4L\omega} - \dfrac{(L - 4M)C_2\omega^2}{4} + j\left\{\dfrac{RC_2\omega}{2} - \dfrac{MRC_2\omega}{L} - \dfrac{R}{L\omega}\right\}}} \tag{142}$$

Equation (142) *gives the exact value of the surge impedance for the line of Type III with resistance in the inductance coil. L and R are the inductance and resistance of each of the coils to the middle of which the capacities C_2 are attached. M is the mutual inductance between the two halves of one coil.*

285. Approximations for Type III.—Out of analogy of the equations in this case with the equations for Type II, and by an examination of the constants in the two cases, it is readily seen that

$$a = \frac{R}{2}\sqrt{\frac{C_2}{L}}, \text{ and } \varphi = \omega\sqrt{LC_2} \tag{143}$$

provided

$$\frac{R^2}{8L^2\omega^2} << 1, \quad \text{and} \quad \frac{LC_2\omega^2}{8(1 + MC_2\omega^2)} << 1 \tag{144}$$

In this case, (142) becomes

$$z_1 = \sqrt{\frac{L}{C_2}} \tag{145}$$

In case of small decrement and small value of $LC_2\omega^2$, as stipulated in (144), *equations* (143) *give the attenuation constant of the current and the retardation angle of the current per section of line with a line of Type III. Under the same conditions the surge impedance of the line is given by* (145).

VIII. COMPUTATION OF APPARATUS

286. Design of a Filter to Cut Out Frequencies Below a Specified Value, and to Operate Between Input and Output Terminal Apparatus of Given Resistance.—For this purpose we require a line of Type I. The coils of such a line will necessarily have certain resistance, and we shall take account of the resistance of these coils in the computation. The equation for the attenuation constant is (108). This expression for a begins to increase rapidly in the neighborhood of the value of θ at which $1 - 4\theta$ becomes negative, with increasing θ.

We shall call the value of ω at which

$$4\theta = 1 \tag{146}$$

the cut-off value of angular velocity.

Now in general

$$\theta = L_2C_1\omega^2 \tag{147}$$

Let ω_0 = angular velocity below which the filter is to give high attenuation.

Then by (147) and (146), we must make L_2 and C_1 such that

$$\omega_0 = \frac{1}{2\sqrt{L_2C_1}} \tag{148}$$

Equation (148) *gives one relation for determining L_2 and C_1 to comply with the cut-off requirement.*

We shall next find another relation determined by the resistance of the terminal apparatus. To avoid reflection the complex impedance z_0 of the input apparatus and the complex impedance z_T of the output apparatus shall each equal the surge impedance of the line, which is z_i; that is

$$z_0 = z_T = z_i \tag{149}$$

Now the value of z_i for this type of line is given in (110), and is a complicated function of the frequency. We cannot in general make z_i equal to z_0 and z_T for all values of the frequency.

Let it be supposed that while we wish to cut off all frequencies of angular velocity less than ω_0, we are also interested in transmitting especially the high frequencies for which the conditions (118) are satisfied. For these frequencies

$$z_i = \sqrt{L_2/C_1} \tag{150}$$

and is in the nature of a pure resistance independent of the frequency. We should need to make our terminal apparatus as nearly as possible a pure resistance, of value

$$R_0 = \sqrt{L_2/C_1} \tag{151}$$

where

R_0 = resistance of input apparatus and of output apparatus, which are to be nearly pure resistances

Equation (151) is another relation for determining L_2 and C_1, and is obtained on the assumption that the line is to be non-reflective at the terminal apparatus for high frequencies.

Elimination between (151) and (148) gives as the required constants of the line

$$L_2 = R_0/2\omega_0, \quad \text{and } C_1 = 1/2R_0\omega_0 \tag{152}$$

Equations (152) give the value of the inductance and capacity elements of the line to cut off angular velocities above ω_0 and to operate between an input terminal apparatus of resistance R_0 (inductanceless) and an output terminal apparatus of the same resistance.

Now as to the resistance of the inductance coils used in the line, it is desirable to have this resistance R_1 as low as possible, consistent with space available and cost. Let us suppose that the coils are wound of wire of such size as to give

$$R_2/L_2 = 2\Delta \text{ (say)} \tag{153}$$

Assuming this value, and making preparation to employ (108) to determine the performance of the computed filter, let us note that by (105)

$$\left.\begin{aligned} \eta_2 &= R_2/L_2\omega = \frac{2\Delta}{\omega_0}\frac{\omega_0}{\omega} \\ \theta &= \frac{\omega^2}{4\omega_0^2} \end{aligned}\right\} \tag{154}$$

As soon as we specify the ratio of Δ to ω_0, we can compute a and φ by (108) and (109) for various ratios of ω to ω_0.

Let us now compute a numerical example, given

$$2\Delta = 250 \text{ and } \omega_0 = 5000 \tag{155}$$

This means that the coils L_2 have 250 ohms per henry, and that we wish to cut off angular velocities below 5000 radians per second.

The results are given in Table IV.

Table IV.—Performance of a Filter Computed to Cut Off all Angular Velocities Less Than ω_0 = 5000. Given R_2/L_2 = 250

ω/ω_0	a	φ (radians)	ϵ^{-10a}
0 2	4 61	0 250	0 0000000000000000001
0 4	3 22	0 136	0 0000000000002
0 6	2 42	0 104	0 00000000003
0.8	1 386	0 104	0 000002
0 9	0 956	0 127	0 000068
0 95	0 661	0 164	0 00136
1 00	0 311	0 322	0 044
1 05	0.138	0 583	0 25
1.10	0 0981	0 868	0 374
1.20	0 0644	1 175	0 525
1.40	0 0364	1 546	0 694
1 60	0 0250	1 351	0 778
2 00	0 0144	1 047	0 865
2 50	0 00873	0 823	0 916
3 00	0 00589	0 680	0 942
4 00	0 00323	0 505	0 967

287. Design of a Compensator to Give a Retardation Time of a Constant Amount T Seconds per Section Substantially Independent of the Frequency over a Significant Range of Frequencies, and to Operate with Terminal Apparatus of Given Resistance.—For this purpose we shall use a line of Type II. In equation (131) it is shown that for ranges of ω for which (132) is fulfilled, the angle of retardation per section is

$$\varphi = \omega\sqrt{L_1C_2},$$

whence

$$T = \varphi/\omega = \sqrt{L_1C_2} \tag{156}$$

where

T = time lag in seconds per section of the line introduced into the current by the line.

The discussion of T is found in equation (97) and thereabouts.

The value of T given in (165) is correct only provided ω is sufficiently removed from the cut-off frequency which we shall specify as having the angular velocity ω_0.

The angular velocity of the cut-off frequency is the value of ω at which the last brace under the outer radical of (124) changes sign. This is approximately the value of ω at which

$$\Psi = 4 \tag{157}$$

or by (121)

$$\omega_0 = 2/\sqrt{L_1C_2} \tag{158}$$

where

ω_0 = angular velocity of cut-off frequency, which is the angular velocity above which the currents are highly attenuated.

It is to be noted that we can determine the product of L_1C_2 either by specifying the desired time lag T per section and using (156), or by specifying the cut-off angular velocity ω_0 and using (158). If we proceed by specifying T, we must make T small enough to raise the angular velocity of the cut-off frequency to give the operating range of frequency required of the apparatus.

The next step in settling upon the essential constants of the compensator is the choice of the impedance of the terminal apparatus. The impedance of the input apparatus z_0, the impedance of the output terminal apparatus z_T and the surge impedance z_t of the line must be equal to avoid reflections in the line, and to obtain a maximum transfer of energy to the output apparatus. If we operate the line in the region of frequencies in which (156) holds, then by (133)

$$z_t = \sqrt{\frac{L_1}{C_2}} = z_0 = z_T = R_0 \text{ (say)} \tag{159}$$

The several impedances in (159) being equal to the radical expression are real quantities independent of the frequency, and must be of the nature of pure resistances.

R_0 = resistance of the input apparatus and of the output apparatus, which must be both inductanceless to avoid reflection.

It may not be possible to utilize terminal apparatus of the nature of pure resistances and attain the results desired. In that case, we can not avoid reflections at the junction of the line with the terminal apparatus, and we shall sometimes need to make a compromise in practice. We shall not here enter into the nature of a profitable compromise, but shall proceed on the assumption that (159) may be fulfilled.

Now eliminating between (159) and (156), we obtain

$$L_1 = R_0T, \quad C_2 = T/R_0 \tag{160}$$

Equations (160) give the inductance and capacity per section of a line of Type II, designed to give a time-retardation of current by the amount T seconds per section, and designed to operate between non-inductive input apparatus of resistance R_0 and non-inductive output apparatus of the same resistance.

By equation (148) this line, if its elements have sufficiently low resistance, will let through with small attenuation all frequencies of angular velocity less than

$$\omega_0 = \frac{1}{2\sqrt{L_1C_2}} = \frac{1}{2T} \tag{161}$$

To compute the performance of such a line we need to specify T and also to specify the resistance R_1 of the inductance coils, but this need be done merely by specifying the ratio of R_1 to L_1.

We give in Table V, the computation of the performance of such a compensator with the specific values.

$$T = 6.5 \times 10^{-5} \text{ seconds, and } \frac{R_1}{L_1} = 250.$$

Table V.—Performance of a Compensator Computed to Give a Time-lag of $T = 6.5 \times 10^{-5}$ Seconds per Section. Given $R_1/L_1 = 250$

ω	n	a	φ	T seconds	ϵ^{-10a}
770	123	0 00802	0 0505	6 55$\times 10^{-5}$	0 923
1,540	245	0 00812	0 100	6 50	0 922
3,080	490	0 00812	0 201	6 52	0 922
6,160	980	0 00825	0 403	6 55	0 921
9,240	1,470	0 0085	0 607	6 60	0 920
12,320	1,960	0 0088	0 825	6 75	0 916
15,400	2,451	0 0098	1.06	6 88	0 906
18,500	2,944	0 0101	1 29	6 97	0 903
21,600	3,438	0 0114	1 54	7 14	0 892
24,640	3,922	0 0135	1.86	7 54	0 873
27,720	4,412	0 0185	2.25	8 04	0 831
29,300	4,660	0 0258	2 54	8 70	0.773
30,800	4,902	0 127	3 01	9 80	0 281
30,954	4,927	0 133	3 02	9 80	0 264
31,108	4,951	0 171	3 05	9 80	0 180
31,416	5,000	0 216	3 07	9 80	0 115
33,880	5,392	0 339	3.09	9 15	0 033
36,960	5,883	0 487	3 10	8.40	0.007
38,500	6,128	0 837	3 12	8 05	0 0002

In the first column of Table V is the angular velocity of the current, which is determined by the angular velocity of the

impressed e.m.f. The second column contains the frequency n corresponding to the given values of ω. The third column contains the attenuation constant per section. The fourth column contains the retardation angle per section. The fifth column contains the time-retardation per section of the line. The last column contains the ratio of the current in the tenth section to the current in the zeroth section.

Notice that the time-retardation per section changes only about one per cent. in the range of frequencies between $n = 123$ and $n = 1470$. Over this range of frequencies the line can be used to introduce known amounts of time-lag by introducing various numbers of sections of the line. The attenuation for ten sections of the line in this range is slight since over 90 per cent. of the current gets through.

As we pass up to higher frequencies, the time-lag per section changes considerably.

At $n = 4902$ the cutting-off effect of the line begins to make its appearance, and at $n = 5883$, the current in the tenth section is less than one per cent. of the current in the zeroth section.

It is to be noted that by making T smaller, the time-lag per section can be made nearly constant for higher frequencies than those given in this table. In fact by making T sufficiently small this compensator action, by which is meant the introduction of time-retardation substantially independent of the frequency, can be made applicable to the ordinary ranges of radio frequency.

CHAPTER XVII

ELECTRIC WAVES ON WIRES IN A STEADY STATE

288. Two Methods.—There are two possible methods of treating the propagation of electric currents along wires; namely:

I. By considering the wires as a limiting case of an electrical system with recurrent similar sections,[1] utilizing the facts obtained in Chapter XVI;

II. By building up directly the differential equations for the currents on the wires and solving the equations anew.[2]

We shall employ the former of these methods. We shall treat the problem only for the steady-state condition.

289. Diagram, Notation, and Impedances.—Referring to Fig. 1, suppose that we have two parallel wires, with a source of e.m.f. at e, having a complex impedance z_0, and with an output apparatus at T, having a complex impedance z_T, let it be required to find the current i at any time t and at any distance x from the e.m.f.

The wires have certain resistance, and inductance, per element of length, and they have a certain capacity per element of length.

Let there be a certain current i flowing out through the top wire at a distance x from the e.m.f., and, on account of symmetry, let there be an equal current in the opposite direction in the lower wire at the same distance x from the e.m.f.

As in Fig. 2, let us divide the wire into lengths Δx, and for each length Δx let us suppose a capacity C_2 between the wires.

[1] For an infinite line this method was employed by E. P. Adams, *Proc. Am Philosophical Soc*, 49, 1910.

[2] This method was employed in a special case by Sir William Thomson (Lord Kelvin) in an examination of the feasibility of the Atlantic Cable in 1855, published in *Proc Roy. Soc.*, May, 1855. The general problem of waves on wires was first treated by Kirchhoff, *Pogg. Ann.*, 100, 1857. Further extensive work on the subject was done by Heaviside, *Phil. Mag*, 1876, and *Electrical Papers*, Vol 1, p. 53.

The wire is thus divided into elemental sections of length Δx. The shunt capacity per section is then,

$$C_2 = C\Delta x \tag{1}$$

where C = capacity per unit of length of the wires.

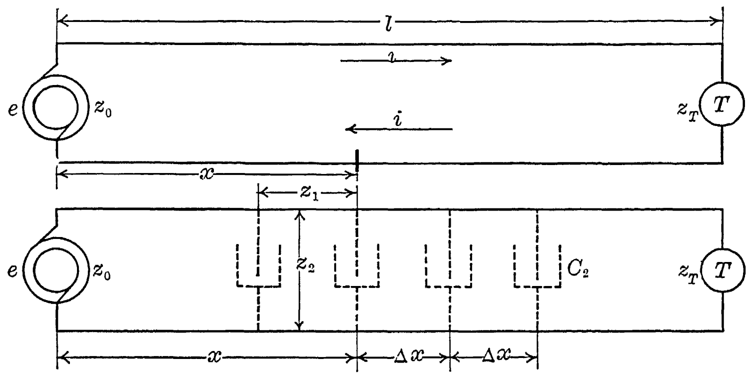

FIG. 1 —Two parallel wires FIG 2.—Resolution of two parallel wires into a system of elemental sections

Assuming that there is no current leakage between the wires, and designating the complex shunt impedance per section of the system as z_2, we have

$$z_2 = -j/C\omega\Delta x \tag{2}$$

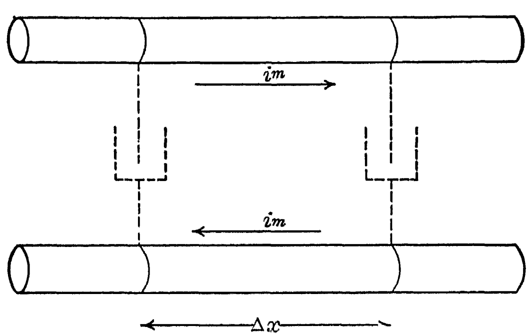

FIG. 3.—The *m*th section of two parallel wires.

Treating the line as made up of sections of length Δx, equation (2) gives the complex shunt impedance per section, provided there is no current leakage between the wires.

Let us consider next the series impedance in each of the sections. A typical section is shown in Fig. 3. If we call this

section the mth section, a current i_m flows in the parts of this section not common to the next sections; that is to say, this current flows in each of the wires, as shown in Fig. 3.

The complex series impedance of this section is

$$z_1 = R\Delta x + jL\omega\Delta x \tag{3}$$

where

R = resistance per loop unit of length = the resistance per unit length of outgoing conductor + resistance per unit of length of return conductor;

L = inductance per loop unit of length = inductance of the two wires per unit length of the duplex system, when one of the wires is a return conductor for the other.

Equation (3) *gives the complex series impedance per section of length* Δx.

Here we may note one other simple relation. If x is the distance from the e.m.f. to the mth section, then

$$x = m\Delta x \tag{4}$$

290. Attenuation Constant and Retardation Angle per Loop Unit of Length of the Wires.—The system of Fig. 2 is an example of a line of Type II of Chapter XVI, and has the attenuation constant and retardation angle per section (that is, per length Δx) given in (124) and (125), Chapter XVI, in which by (121), Chapter XVI, and (3) and (2) of the present chapter,

$$\eta_1 = R/L\omega, \qquad \psi = LC\omega^2(\Delta x)^2 \tag{5}$$

Introducing these values into (124) and (125) of Chapter XVI, and calling the resultant quantities Δa and $\Delta\varphi$, since they are values per length Δx, we obtain

$$\Delta a = \sinh^{-1}\left\{\omega\Delta x\sqrt{\frac{LC}{2}}\sqrt{\sqrt{\frac{R^2}{L^2\omega^2}+1}-1}\right\} \tag{6}$$

$$\Delta\varphi = \sin^{-1}\left\{\omega\Delta x\sqrt{\frac{LC}{2}}\sqrt{\sqrt{\frac{R^2}{L^2\omega^2}+1}+1}\right\} \tag{7}$$

In deriving these equations, we have omitted within the radical terms added to unity and having a factor Δx, because we are going to make Δx approach zero, and such terms would ultimately disappear. The external multiplier Δx we keep, because the

quantities Δa and $\Delta\varphi$ which appear on the left-hand sides of the equations are increments of the same order as Δx.

If we now look at equation (28), Chapter XVI, we shall notice that the complex current amplitude in the mth section contains factors of the form ϵ^{-km}, where by (40) of Chapter XVI

$$k = a + j\varphi = \Delta a + j\Delta\varphi,$$

in the present case of small sections. Hence, employing (4), we obtain

$$\epsilon^{-km} = \epsilon^{-m\Delta a}\epsilon^{-jm\Delta\varphi} = \epsilon^{-\alpha x}\epsilon^{-j\beta x} \tag{8}$$

where

$$\alpha = \frac{\Delta a}{\Delta x}, \qquad \beta = \frac{\Delta\varphi}{\Delta x} \tag{9}$$

in which for continuous values of x, we must take the limit of (9) and (8) as Δx approaches zero, giving

$$\alpha = \frac{\partial a}{\partial x}, \qquad \beta = \frac{\partial\varphi}{\partial x} \tag{10}$$

Equations (10) *give the attenuation constant* α *and the angle of retardation* φ *per unit length of the wires.*

In terms of these quantities ϵ^{-km} *taken for a continuous line the form given in* (8), *where* x *is the length from the e.m.f along the wires to the section of the wires under consideration.*

We may now obtain values of α and β from (6) and (7) by dividing by Δx and taking the limit as Δx approaches zero, noting that the antihyperbolic sine and the antisine approach their moduli as Δx approaches zero. By this operation we obtain

$$\alpha = \omega\sqrt{\frac{LC}{2}}\sqrt{\sqrt{\frac{R^2}{L^2\omega^2} + 1} - 1} \tag{11}$$

$$\beta = \omega\sqrt{\frac{LC}{2}}\sqrt{\sqrt{\frac{R^2}{L^2\omega^2} + 1} + 1} \tag{12}$$

Equations (11) *and* (12) *give respectively the attenuation constant and the retardation angle per unit length of the pair of wires.* ω *is the angular velocity of the impressed e.m.f.* $R, L,$ *and* C *are respectively the Resistance, Inductance, and Capacity per loop unit of length.*

291. Approximate Determination of α and β in Special Cases. I. In the range of frequencies in which

$$R/L\omega < 1 \tag{13}$$

we may expand the radicals obtaining

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}}\left\{1 - \frac{R^2}{4L^2\omega^2} + \frac{R^4}{8L^4\omega^4} - \qquad \right\}^{\frac{1}{2}} \tag{14}$$

$$\beta = \omega\sqrt{LC}\left\{1 + \frac{R^2}{4L^2\omega^2} - \frac{R^4}{16L^4\omega^4} + \cdot \quad \cdot\right\}^{\frac{1}{2}} \tag{15}$$

II. In the range of frequencies in which

$$R^2/8L^2\omega^2 << 1 \tag{16}$$

equations (14) and (15) become

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}}, \qquad \beta = \omega\sqrt{LC} \tag{17}$$

III. In the range of frequencies in which

$$R/L\omega > 1 \tag{18}$$

we may expand the radicals in (11) and (12) and obtain

$$\alpha = \sqrt{\frac{RC\omega}{2}}\left\{1 - \frac{L\omega}{R} + \frac{L^2\omega^2}{2R^2} - \frac{L^4\omega^4}{8R^4} + \qquad \right\}^{\frac{1}{2}} \tag{19}$$

$$\beta = \sqrt{\frac{RC\omega}{2}}\left\{1 + \frac{L\omega}{R} + \frac{L^2\omega^2}{2R^2} - \frac{L^4\omega^4}{8R^2} + \cdot \qquad \right\}^{\frac{1}{2}} \tag{20}$$

IV. In the range of frequencies in which

$$\frac{L\omega}{2}/R << 1 \tag{21}$$

$$\alpha = \sqrt{\frac{RC\omega}{2}}, \qquad \beta = \sqrt{\frac{RC\omega}{2}} \tag{22}$$

Equations (14) *and* (15) *give respectively the attenuation constant* α *per unit length of the line and the retardation angle* β *per unit length of the line, provided* (13) *is satisfied.*

If (16) *is satisfied, the corresponding values of* α *and* β *are given by* (17). *Under conditions* (18), α *and* β *are given by* (19) *and* (20) *respectively. Under conditions* (21), *these quantities are given by* (22).

In these equations R, L, and C are respectively the Resistance, Inductance, and Capacity per loop unit of length.

292. Surge Impedance of the Line.—If in (125*a*) of Chapter XVI, we replace R_1 and L_1 by $R\Delta x$ and $L\Delta x$ respectively and neglect terms involving $(\Delta x)^2$ in comparison with unity, we shall

have for the surge impedance z_i of the continuous line, the value

$$z_i = \sqrt{\frac{L}{C}}\sqrt{1 - j\frac{R}{L\omega}} \tag{23}$$

Equation (23) is the exact expression for the surge impedance of the continuous line in which R, L, and C are respectively the Resistance, Inductance, and Capacity per loop unit of length of the line.

This becomes

$$z_i = \sqrt{\frac{L}{C}}, \text{ provided } R^2/2L^2\omega^2 << 1 \tag{24}$$

It becomes

$$z_i = \sqrt{-j}\sqrt{\frac{R}{C\omega}}, \text{ provided } L^2\omega^2/2R^2 << 1 \tag{25}$$

293. Reflection Coefficients. Condition for No Reflection. The complex reflection coefficient X at the input apparatus by (39), Chapter XVI, is

$$X = \frac{z_i - z_0}{z_i + z_0} \tag{26}$$

where z_0 = impedance of input apparatus.

Likewise the complex reflection coefficient Y at the output apparatus is

$$Y = \frac{z_i - z_T}{z_i + z_T} \tag{27}$$

where z_T = impedance of output terminal apparatus.

294. General Expression for the Complex Current Amplitude at a Distance x from the Impressed e.m.f., When the Length of the Parallel Wires from Input Apparatus to Output Apparatus is l.—To obtain this value, we shall use the general equation (28), Chapter XVI, with proper transformation to suit the smooth line problem.

We have already found in (8) of the present chapter that

$$\epsilon^{-km} = \epsilon^{-\alpha x}\epsilon^{-j\beta x}.$$

In this we have made the mth section a distance x from the e.m.f. The total length of the present line is to be l, and the total number of sections of the discrete line of equation (28), Chapter XVI, was n, so that if we replace m by n and x by l, we have by the equation next above

$$\epsilon^{-kn} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} \tag{28}$$

Also in Chapter XVI, equations (30) and (32), we have

$$-b(\epsilon^k - x) = z_0 + z_i \tag{29}$$

Substituting these several values into (28), Chapter XVI, and designating the resulting value of A_m by the A_x, we have

$$\left.\begin{aligned} A_x = \frac{E}{z_0 + z_i}\{\epsilon^{-\alpha x}\epsilon^{-j\beta x} + & \quad Y\epsilon^{-\alpha(2l-x)}\epsilon^{-j\beta(2l-x)} \\ & + XY\epsilon^{-\alpha(2l+x)}\epsilon^{-j\beta(2l+x)} \\ & + XY^2\epsilon^{-\alpha(4l-x)}\epsilon^{-j\beta(4l-x)} \\ & + X^2Y^2\epsilon^{-\alpha(4l+x)}\epsilon^{-j\beta(4l+x)} \\ & + \ldots\ldots\ldots\ldots \} \end{aligned}\right\} \tag{30}$$

In deriving this equation we have assumed that the impressed e.m.f. is

$$e = E\epsilon^{j\omega t} \tag{31}$$

The complex current at x is

$$i_x = A_x\epsilon^{j\omega t} \tag{32}$$

and is valid *only in the steady state.*

Equation (30) *is a general expression for the complex current amplitude A_x at a distance x from the impressed e.m.f., for the case of two parallel wires each of length l, terminated by an output apparatus of impedance z_T connecting the two wires together at their outer end. The input apparatus has impedance z_0 and connects the wires together at their input end. The values of z_i, X, and Y are given in* (23), (26), *and* (27).

295. Real Current for an Infinite Smooth Line or a Line with Non-reflective Output Impedance. Velocity of Propagation. Phase Change by Reflection.—If the line is infinite, or if $Y = 0$, all of the terms after the first in (30) disappear, and if we make the impressed e.m.f.

$$e = E \sin \omega t \tag{33}$$

and take $1/j$ times the imaginary part of (32), in which A_x has been replaced by its value from (30), with all the terms of (30) after the first set equal to zero, we obtain

$$i_x = \frac{E}{\overline{\overline{Z}}}\epsilon^{-\alpha x} \sin \{\omega(t - \frac{\beta x}{\omega}) - \theta'\} \tag{34}$$

where

$$\overline{\overline{Z}} = \sqrt{\overline{\overline{R}}^2 + \overline{\overline{X}}^2}, \quad \theta' = \tan^{-1}\{\frac{\overline{\overline{X}}}{\overline{\overline{R}}}\} \tag{35}$$

The values of $\overline{R}$ and $\overline{X}$ are resistance and reactance, respectively of $z_0 + z_i$; that is,

$$z_0 + z_i = \overline{R} + j\overline{X} \tag{36}$$

Equation (34) *gives the current at distance x from the source of e.m.f. for two parallel wires infinite in extent, or with a non-reflective output impedance.*

The expression (34) may be looked upon as made up of the product of three factors as follows:

$$\frac{E}{\overline{Z}} = \text{amplitude of current at } x = 0.$$

$$\epsilon^{-x\alpha} = \text{attenuation factor,}$$

by which the current-amplitude at $x = 0$ is to be multiplied to get the current-amplitude at $x = x$.

$\sin\{\omega(t-\beta x/\omega)-\theta')$ = the periodic factor, which is periodic in t and periodic in x.

It may be noted that in the periodic factor

θ' = lag of current behind e.m.f. at $x = 0$.

βx = lag of current at $x = x$ behind current at $x = 0$.

We may now obtain the velocity of propagation by noting that the periodic term at $t = t_2$ and $x = x_2$ will have the same value that it has at $t = t_1$ and $x = x_1$, provided

$$t_2 - \beta x_2/\omega = t_1 - \beta x_1/\omega \tag{37}$$

whence

$$\frac{x_2 - x_1}{t_2 - t_1} = \omega/\beta = \text{velocity of propagation} \tag{38}$$

The quantity on the left of (38) is seen to be the velocity of propagation, because $t_2 - t_1$ is the time that must elapse for a given phase of the disturbance to travel from x_1 to x_2, and whatever the values of x_1 and x_2 the ratio of the distance to time is independent of the distance.

Equation (38) *shows that the velocity of propagation of the disturbance along the wires is*

$$v = \omega/\beta \tag{39}$$

where β is given exactly by equation (12), *and is further given in approximate form for special cases in* (15), (17), (20) *and* (22).

Although we derived v on the assumption of a non-reflective line the result is correct for any line, for the terms after the first in (30) *give the same velocity v for each term. We must, however, when X and Y are complex quantities attribute to the reflected waves a change of phase at reflection, which is*

$$\theta_T = \tan^{-1}\left\{\frac{\text{coefficient of } j \text{ in the imaginary part of } Y}{\text{real part of } Y}\right\} \tag{40}$$

and

$$\theta_0 = \tan^{-1}\left\{\frac{\text{coefficient of } j \text{ in the imaginary part of } X}{\text{real part of } X}\right\} \tag{41}$$

Equations (40) *and* (41) *give the angle by which the reflected current lags behind the incident current at the output impedance and the input impedance respectively.*

296. Velocity and Attenuation of High-frequency Waves on Parallel Wires or on Two Concentric Tubes.—The velocity of a sinusoidal current in the steady state on two parallel wires is

$$v = \omega/\beta.$$

By reference to the value of β given in (12) it is seen that v is in general a function of the frequency. But by (16) and (17) it is seen that if ω is sufficiently large to make

$$R^2/8L^2\omega^2 < < 1, \quad \text{then } v = 1/\sqrt{LC} \tag{42}$$

Equation (42) *gives the velocity v of propagation along two parallel wires. The same equation evidently holds for propagation along two tubes, one inside of the other and coaxial with it. In* (42) *L and C are inductance and capacity per loop unit of length, and the unit of length must be the same as the unit of length occurring in the velocity.*

The inductance capacity and velocity must be measured in some consistent set of units.

Formulas for the inductance and capacity of parallel wires and of concentric tubes are well known as follows:

For two parallel wires in which the current is flowing only on the outside surface, one being a return wire,

$$L = \frac{4\mu \log_\epsilon \frac{d}{r}}{c^2} \text{ c.g.s. electrostatic units per centimeter of length of wires} \tag{43}$$

$$= 4\mu \log_\epsilon \frac{d}{r} \text{ c.g.s. electromagnetic units per centimeter of length of wires} \tag{44}$$

$$= \frac{4\mu \log_\epsilon \frac{d}{r}}{10^9} \text{ henries per cm. length} \tag{45}$$

in which

L = inductance per centimeter (loop) of length,
r = radius of one wire,
d = axial distance between wires,
μ = magnetic permeability of the medium between the wires,
c = ratio of electromagnetic unit of quantity to electrostatic unit of quantity = 3×10^{10} cm /sec.

In the same case the capacity is

$$C = \frac{k}{4 \log_\epsilon \frac{d}{r}} \text{ c.g.s. electrostatic units per loop centimeter of length of wires} \tag{46}$$

$$= \frac{k}{4c^2 \log_\epsilon \frac{d}{r}} \text{ c.g.s. electromagnetic units per loop centimeter of length of wires} \tag{47}$$

$$= \frac{k10^9}{4c^2 \log_\epsilon \frac{d}{r}} \text{ farads per loop centimeter of length of wires} \tag{48}$$

where

C = capacity per loop centimeter of length of wires,
k = dielectric constant of the medium between wires,
c = ratio of units = 3×10^{10} cm./sec.

In like manner for two coaxial tubes with the current only on the adjacent surfaces

$$L = \underset{\text{e.s.u.}}{\frac{2\mu \log_\epsilon \frac{R_2}{R_1}}{c^2}} = \underset{\text{e.m.u.}}{2\mu \log_\epsilon \frac{R_2}{R_1}} = \underset{\text{henries}}{\frac{2\mu \log \frac{R_2}{R_1}}{10^9}} \text{ per loop centimeter of length} \tag{49}$$

$$C = \underset{\text{e.s.u.}}{\frac{k}{2 \log_\epsilon \frac{R_2}{R_1}}} = \underset{\text{e.m.u.}}{\frac{k}{2c^2 \log_\epsilon \frac{R_2}{R_1}}} = \underset{\text{farads}}{\frac{k10^9}{2c^2 \log_\epsilon \frac{R_2}{R_1}}} \text{ per loop centimeter of length} \tag{50}$$

in which

R_2 = inner radius of outer cylinder,
R_1 = outer radius of inner cylinder.

By taking the square root of the product of L and C in any one of the sets of units, for the case of the parallel wires or for the case of the coaxial tubes, we obtain by (42)

$$v = c/\sqrt{k\mu}, \text{ provided } R^2/8L^2\omega^2 < < 1 \tag{51}$$

Equation (51) gives the velocity of propagation of high-frequency waves on two parallel wires or on two coaxial tubes. In this equation c, which is the ratio of the electromagnetic unit of quantity to the electrostatic unit of quantity, has been shown by experiment to be equal to the velocity of light. If the medium between the wires is a vacuum $k = \mu = 1$, and

$$v = c \tag{52}$$

that is, the velocity of the high-frequency waves on parallel wires or coaxial tubes is equal to the velocity of light, when the medium around the wires or between the tubes has dielectric constant and permeability unity.[1]

As to the attenuation constant in this case of high-frequency waves, a substitution of C in farads and L in henries per unit length into (17) for α gives

$$\alpha = \frac{R}{2}\sqrt{\frac{k}{\mu}}\frac{1}{120\log_\epsilon \frac{d}{r}} \text{ for parallel wires} \tag{53}$$

[1] Direct experimental determinations of the velocity of high-frequency waves on wires have been made as follows:

Observer	Velocity in centimeters per second	Published in
Blondlot .	2 930 $\times 10^{10}$ 2 980	*Comp. Rend.*, 117, p. 543, 1893
Trowbridge and Duane . . .	2 980 3 003	*Am. Journ. of Sci*, 49, p. 297, 1895.
Saunders.......	2 954 2.994 2 998 2 998 2 995 2 999	*Phys. Rev.*, 4, p. 81, 1896.

For best determinations of the velocity of light see Book II, Chapter IV, Art. 42.

and

$$\alpha = \frac{R}{2}\sqrt{\frac{k}{\mu}}\frac{1}{60\log_\epsilon\frac{R_2}{R_1}} \text{ for coaxial tubes} \tag{54}$$

where k, μ, d, R, R_1 and R_2 have values given above.

Equations (53) and (54) give the attenuation constants per loop unit of length for two parallel wires and for two coaxial tubes respectively. In these equations R is the resistance in ohms per loop unit of length, using the same unit of length that is applied to the attenuation constant.

These equations apply only to cases of sufficiently high frequency to make $R^2/8L^2\omega^2$ negligible in comparison with unity.

297. Stationary High-frequency Waves on Two Parallel Wires Open-ended at Outer End and Non-reflective at Input End.—Reference is made to Fig. 4. Let the length of one wire

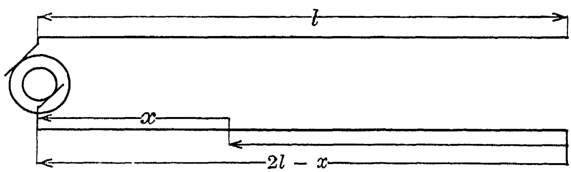

FIG. 4.—Showing direct and reflected distances.

from the e.m.f. to the open end be l. The open end is equivalent to an infinite terminal resistance. Therefore by (27)

$$Y = -1 \tag{55}$$

We shall now make the input impedance non-reflective, which by (26) and (24) gives

$$\left.\begin{array}{l} X = 0;\, z_0 = z_i = \sqrt{L/C}, \\ \text{provided} \\ R^2/2L^2\omega^2 << 1 \end{array}\right\} \tag{56}$$

Also referring to (35) and (36) we have

$$\overline{Z} = 2\sqrt{L/C}, \text{ and } \theta' = 0 \tag{57}$$

If now

$$e = E \sin \omega t \tag{58}$$

and we take the sine part of (30), with attention to (55), (56) and (57), we obtain

$$i_x = \frac{E}{2\sqrt{L/C}}\{\epsilon^{-\alpha x} \sin [\omega(t - x/v)] - \epsilon^{-\alpha(2l-x)} \sin [\omega(t - (2l - x)/v)]\}$$

where

$$v = c/\sqrt{k\mu} \tag{60}$$

k and μ = dielectric constant and permeability of medium around the wires, and where

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \tag{61}$$

Equation (59) gives the steady-state current at the distance x from the e m.f. for the case of high-frequency waves on two parallel wires of length l open-ended at the outer end and non-reflective at the input end. The current is seen to be the resultant of two wave-systems —one passing direct from the source of e.m.f., and the other reflected with a reversal of sign from the open end of the system. The out-going wave has traveled a distance x and the reflected wave has traveled a distance $l + l - x$.

It is to be noted that at the outer end of the wires, where $x = l$, equation (59) gives $i = 0$.

On the other hand, at $x = 0$, the current is

$$i_0 = \frac{E}{2\sqrt{L/C}}\{\sin \omega t - \epsilon^{-2\alpha l} \sin \omega(t - 2l/v)\} \tag{62}$$

Equation (62) gives the current at the input end of a line with non-reflective input impedance and with outer end of the line open.

From equation (62) it may be noted that if $\epsilon^{-2\alpha l}$ is nearly equal to unity, we shall get the largest value of i_0, if we make the length of the line such that the second term is brought into phase with the first term; that is, if

$$2\omega l/v = \pi,\ 3\pi,\ 5\pi,\ \ldots \tag{63}$$

If we multiply numerator and denominator of (63) by T, the period of the e.m.f., and note that

$$\omega T = 2\pi, \text{ and } vT = \lambda_1$$

where λ_1 = the wavelength of the waves on the wires, we have, as the condition for a maximum value of i_0,

$$l = \lambda_1/4,\ 3\lambda_1/4,\ 5\lambda_1/4,\ \ldots \tag{64}$$

When the attenuation factor $\epsilon^{-2\alpha l}$ is nearly equal to unity, we obtain a maximum amplitude of current at the input end when the length of each of the wires is an odd number of quarter wavelengths of the waves on wires; provided the outer end of the system is open-

ended, and provided the input impedance is non-reflective and is excited by a high-frequency sinusoidal e.m.f.

298. Stationary High-frequency Waves on Two Parallel Wires Non-reflective at the Input End and Terminated by a Condenser C′ at Outer End.—Reference is made to Fig. 5.

The output terminal impedance in this case is

$$z_T = -j/C'\omega \tag{65}$$

The input impedance and the surge impedance of the line are

$$z_0 = z_i = \sqrt{L/C} = R_0 \text{ (say)} \tag{66}$$

where L and C are inductance and capacity per loop unit of length of the wires. Equation (66) in the condition for non-reflection at input end

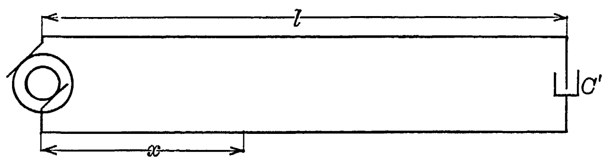

FIG. 5.—Parallel wires connected at outer end through a condenser C'.

Introducing (65) and (66) into (27) we have for the reflection coefficient at the outer end of the line

$$Y = \frac{R_0 + j/C'\omega}{R_0 - j/C'\omega} = \epsilon^{2j \tan^{-1}(1/R_0C'\omega)} \tag{67}$$

The last step is taken by the principles of Chapter IV. Introducing these values into (30) and passing to the case of

$$e = E \sin \omega t \tag{68}$$

we have

$$i_x = \frac{E}{2R_0}\Big\{\epsilon^{-\alpha x} \sin \omega(t - x/v) + \epsilon^{-\alpha(2l-x)} \sin \left[\omega(t - (2l - x)/v) + 2 \tan^{-1} \frac{1}{R_0C'\omega}\right]\Big\} \tag{69}$$

Equation (69) *gives the current at x under the conditions stipulated in the caption.*

If we make $x = 0$, we obtain for the current

$$i_0 = \frac{E}{2R_0}\left\{\sin \omega t + \epsilon^{-2\alpha l} \sin \left[\omega(t - 2l/v) + 2 \tan^{-1} \frac{1}{R_0C'\omega}\right]\right\} \tag{70}$$

We may call the system resonant with the angular velocity ω, when the length of the wires or when the capacity C' is so adjusted as to give a maximum amplitude of the current i_0.

When $\epsilon^{-2\alpha l}$ is nearly unity, i_0 is a maximum if

$$-2\tan^{-1}\frac{1}{R_0C'\omega}+\frac{2l\omega}{v}=0,\ 2\pi,\ 4\pi,\ 6\pi,\ .\ .\ . \tag{71}$$

Equation (71) *gives a series of relations among C', l, and ω, which are proper relations to make i_0 a maximum with the system of circuits shown in Fig.* 5, *consisting of two parallel wires terminated at their outer end by a bridging condenser C' and having the e.m f. applied through an impedance that is non-reflective with respect to the line.*

299. Examination of the Resonant Fundamental System of the Type of Fig. 5.—As an introduction to the general subject of distributed capacity in coils, we shall examine further the system shown in Fig. 5 with reference to its adjustment for *fundamental resonance* with the impressed e.m.f. By fundamental resonance we shall mean the resonance that gives a maximum amplitude of current at $x = 0$ without any other maximum amplitude of current along the wires. This is to be distinguished from harmonic resonance in which there will be a series of current maxima between the e.m.f. and the condenser.

At fundamental resonance, the quantities satisfy (71) with the right-hand side set equal to zero, so that

$$\frac{l\omega}{v}=\tan^{-1}\left\{\frac{1}{R_0C'\omega}\right\} \tag{72}$$

Taking the tangent of both sides of this equation, and replacing R_0 by its value from (66) into which v is introduced from (42), we obtain

$$\tan\frac{l\omega}{v}=\frac{C}{C'\omega\sqrt{LC}}=\frac{Cv}{C'\omega} \tag{73}$$

whence

$$\frac{l\omega}{v}\tan\frac{l\omega}{v}=\frac{Cl}{C'} \tag{74}$$

Equation (74) *gives the relation between the attached condenser C', the length of the parallel wires l, and the impressed angular velocity ω that must be fulfilled to give the maximum current amplitude at the e.m.f. for the fundamental adjustment of a system of the form of Fig.* 5, *actuated by a high-frequency e.m.f.*

As an approximation, let us expand the tangent by the formula

$$\tan y = y + \frac{y^3}{3} + \frac{2y^5}{15} + \frac{17y^7}{315} + \quad . \left[y^2 < \frac{\pi^2}{4}\right]$$

and take the reciprocal of (74) obtaining

$$C' = Cl\left\{\frac{1}{1 + \frac{1}{3}\left(\frac{l\omega}{v}\right)^2 + \frac{2}{15}\left(\frac{l\omega}{v}\right)^4 + \ . \ . \ . \ .}\ \frac{v^2}{l^2\omega^2}\right\}$$

$$= Cl\left\{\frac{v^2}{l^2\omega^2} - \frac{1}{3}\right\}, \text{ provided } \frac{3}{135}\left(\frac{l\omega}{v}\right)^4 << 1 \qquad (75)$$

If now we wish to express this result in terms of the wave-length λ in free space of a wave of angular velocity ω, we may write

$$\omega = \frac{2\pi c}{\lambda} \qquad (76)$$

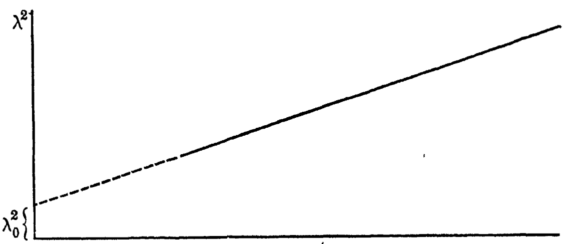

FIG. 6.—Linear relation of λ^2 to C'.

whence (75) becomes

$$C' = Cl\left\{\frac{v^2}{l^2}\frac{\lambda^2}{4\pi^2c^2} - \frac{1}{3}\right\}.$$

By transposition, and replacing Cv^2 by $1/L$, we obtain

$$\lambda^2 = 4\pi^2c^2(lLC') + 4\pi^2c^2\left(\frac{l^2LC}{3}\right) \qquad (77)$$

provided

$$\frac{1}{45}\left\{\frac{2\pi c\sqrt{LCl^2}}{\lambda}\right\}^4 << 1 \qquad (78)$$

Equations (77) and the inequality (78) may also be written

$$\lambda^2 - \lambda_0^2 = B^2C', \text{ provided } \frac{1}{5}\left(\frac{\lambda_0}{\lambda}\right)^4 << 1 \qquad (79)$$

where

$$B^2 = 4\pi^2c^2lL, \quad \text{and } \lambda_0^2 = B^2lC/3 \qquad (80)$$

Equation (79) *gives the capacity C' that must be placed at the outer end of two parallel wires each of length l, to bring the system to resonance with an impressed e.m.f. whose wavelength in free space is λ. This equation applies accurately provided the condition stipulated in* (79) *is met. If various values of λ^2 and the corresponding values of C' with fixed value of l are plotted the result is a straight line of the form of Fig.* 6.

300. Approximate Application to a Coil of Distributed Capacity. The result obtained in the form of (79) for the condition under which a system of two parallel wires with a condenser at the outer end is resonant to an impressed e.m.f., is found by experiment[1] to hold approximately for a coil attached to a condenser as in Fig. 7.

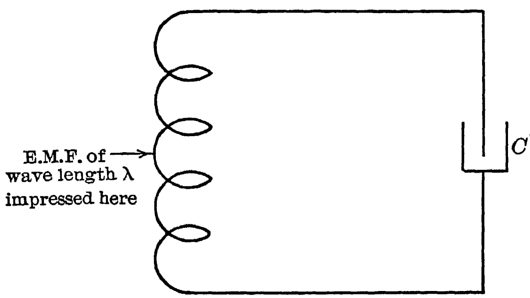

FIG. 7—Coil and condenser.

If we apply to the coil an e.m.f. near its middle section, as may be done by induction from another oscillating circuit, and if we give to the impressed e.m.f. various wavelengths λ, and resonate by giving the condenser C' various values of capacity, it is found that an approximate relation in the form of (79) holds, in that λ^2 minus a constant $\lambda_0{}^2$ is proportional to C', and the plot of the result is similar to Fig. 6.

This result can be accounted for by attributing to the coil a capacity per unit length and an inductance per unit length (of wire or of axial length) provided the product of these quantities is constant for different sections of length. It is not believed that this is exactly the case, but is true to the degree of approximation to which the linear relation of λ^2 to C' is true.

[1] J. C. Hubbard, "On the Effect of Distributed Capacity in Single Layer Solenoids," *Phys. Rev.*, 9, p 529–541, 1917.

301. Difference of Potential Between Two Parallel Wires in Relation to Current Distribution Along the Wires.—Returning now to the general problem of the transmission of electric disturbances along two parallel wires, we may note the following general relations that are true whatever the terminal conditions of the wires and whether the currents are in a steady state or not. We omit only from consideration the cases in which there is leakage of current across from one wire to the other in the region of length under consideration

Reference is made to Fig. 8. Let x be a distance along the wires measured from some arbitrary origin. Let Δx be an element of length at x. Let i be the current flowing into the ele-

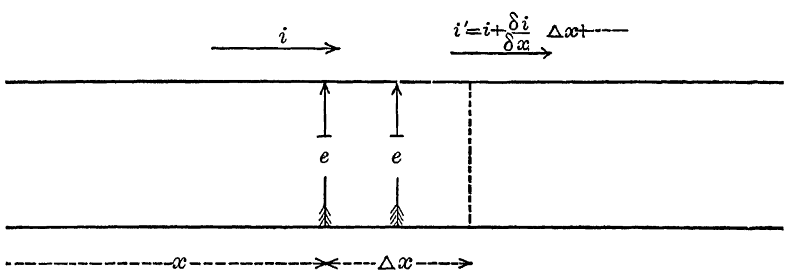

FIG 8 —Used to obtain relation of e to i.

ment Δx at any time t, where i is some function of x and t; that is

$$i = i(x, t) \tag{81}$$

where the i on the right-hand side indicates a functional relation.

Equation (81) *is a formal expression for the current flowing into the section* Δx *at* x *and* t.

To get an expression for the current flowing out of Δx, we need merely note that this current is at a distance $x + \Delta x$ from the origin, and write

$$i' = i(x + \Delta x, t),$$

which expanded by Taylor's Theorem gives

$$i' = i + \frac{\partial i}{\partial x} \Delta x + \ . \ . \ . \tag{82}$$

where the dots represent terms of higher order in Δx.

Equation (82) *is a formal expression for the current flowing out of* Δx.

If now we let $\bar{e}$ be the average excess of the potential of the top wire over the potential of the bottom wire and note that the capacity of a length Δx of the top wire is $C\Delta x$, we have for the charge on the top wire in the element of length Δx the value

$$\Delta q = \bar{e}C\Delta x \tag{83}$$

Now by Kirchhoff's current law the excess of current flowing into Δx over the current flowing out is the time rate of increase of charge of Δx; that is

$$i - i' = \frac{\partial(\Delta q)}{\partial t} \tag{84}$$

The substitution of (81), (82) and (83) into (84) gives

$$-\frac{\partial i}{\partial x}\Delta x + \cdot \quad \cdot = C\Delta x\frac{\partial \bar{e}}{\partial t}.$$

Dividing this equation by Δx and taking the limit as Δx approaches zero, and noting that the terms of higher order in Δx disappear, and that the average value of e in the region approaches the actual value e at x, we obtain

$$-\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t} \tag{85}$$

Equation (85) is an important differential equation connecting the current i at any distance x at any time t with the difference of potential e between the wires at the same x and t.

By continuing this process or reasoning, and applying also Kirchhoff's e.m.f. law to the element of length Δx of both wires, we can build up completely the proper differential equations for the waves on wire and obtain all of the results obtained above by the other method. We shall not do this, but shall merely make application of equation (85) to a single case.

302. Distribution of Current and Potential Along Two Parallel Wires, with the Outer End Open, and with a Non-reflective Input Impedance, Assuming Negligible Attenuation.—Circuits for this case are given in Fig. 4. If the attenuation constant α is negligible, the current may be obtained from (59) by replacing the exponentials by unity. This gives

$$i = \frac{E}{2\sqrt{L/C}}\{\sin\omega(t - x/v) - \sin\omega[t - (2l - x)/v]\} \tag{86}$$

Substituting this value of i into (85), we obtain

$$\frac{\partial e}{\partial t} = -\frac{1}{C}\frac{\partial i}{\partial x}$$
$$= \frac{1}{C}\frac{\omega}{v}\frac{E}{2\sqrt{L/C}}\{\cos\omega[t - x/v] + \cos\omega[t - (2l - x)/v]\}.$$

Integrating this equation with respect to t and replacing v by its value $1/\sqrt{LC}$, we obtain

$$e = \frac{E}{2}\{\sin\omega[t - x/v] + \sin\omega[t - (2l - x)/v]\} \tag{87}$$

Equations (86) *and* (87) *are the values of current and potential at distance x from the origin at time t, with the electrical system shown in Fig.* 4.

Let us next take the special case in which the amplitude of current on the wires is a maximum. By (63) and (64) this is the case in which the length of wires l satisfies the equation

$$\left.\begin{aligned} l &= \lambda_1/4, \quad 3\lambda_1/4, \quad 5\lambda_1/4, \ldots, \\ \text{or} \qquad \omega l/v &= \pi/2, \quad 3\pi/2, \quad 5\pi/2, \ldots \end{aligned}\right\} \tag{88}$$

In this case (86) and (87) become

$$i = \frac{E}{2\sqrt{L/C}}\{\sin\omega(t - x/v) + \sin\omega(t + x/v)\} \tag{89}$$

$$e = \frac{E}{2}\{\sin\omega(t - x/v) - \sin\omega(t + x/v)\} \tag{90}$$

By expanding the sines of the sum and difference terms and collecting, these equations become

$$i = \frac{E}{\sqrt{L/C}}\sin\omega t\cos\frac{\omega x}{v} \tag{91}$$

and

$$e = -E\cos\omega t\sin.\frac{\omega x}{v} \tag{92}$$

Equations (91) *and* (92) *give the current and potential along two parallel wires of length an odd number of times the quarter wavelength of the waves on the wires, provided the outer end of the wires is open, and provided the e.m.f. is impressed through a non-reflective impedance at the input end. The current and potential are out of phase with each other in time and space.*

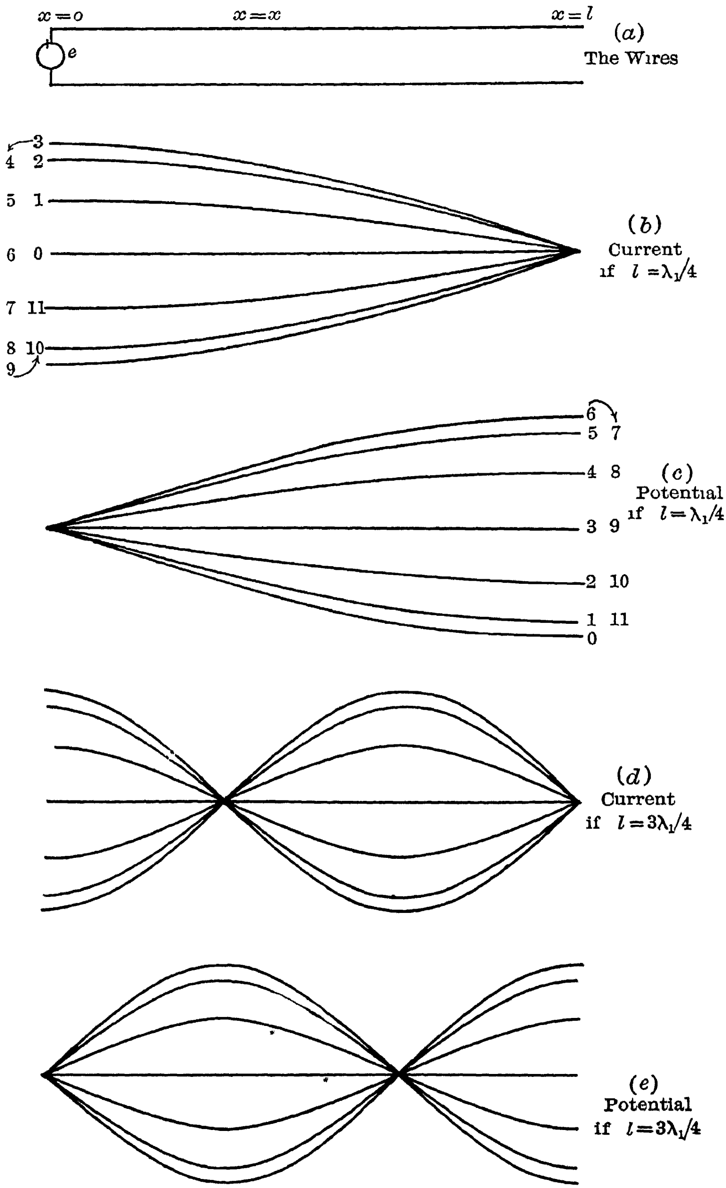

FIG 9.—Stationary waves on wires.

303. Plot of Stationary Current and Potential Waves on Wires of §302.—A plot of equations (91) and (92) for two different cases is given in Fig. 9. In this figure (*a*) represents the wires; (*b*) represents the current distribution along the wires if the wires are $\frac{1}{4}$ wavelength long; (*c*) represents the potential distribution in that case. The curves (*d*) and (*e*) show respectively the current distribution and the potential distribution if the length of each of the wires is $\frac{3}{4}$ of a wavelength.

In each of the diagrams the different curves correspond to different times. For example, in (*b*) and (*c*) these curves are numbered 0 to 11. The curves numbered 0 in the two diagrams are respectively the current and potential at $t = 0$. The curves numbered 1, 2, 3 . . . show the values of current and potential at times equal to $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$ . . . of a whole period after $t = 0$

Book II

ELECTRIC WAVES

CHAPTER I

ELECTROSTATICS AND MAGNETOSTATICS

1. Electric Intensity.—In a field of electric force the force is said to have at every point a certain *intensity*, which is defined as the force with which a unit positive charge of electricity would be impelled if introduced at the point without changing the existing distribution of force. In order not to change the existing distribution the exploring charged body must be a very small body with a very feeble charge, and the force per unit charge is obtained by dividing the force by the charge.

The electric intensity is a *vector*, which we shall designate by $\mathbf{E}$ in Clarendon Type. Throughout this volume all vectors shall be designated by heavy-faced, or Clarendon, type; all scalars by light-faced type. The vector components of $\mathbf{E}$ in the directions x, y, z shall be designated by $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$. The scalar magnitude of $\mathbf{E}$ shall be designated by E with components E_x, E_y, and E_z; unit vectors along the axes of x, y, z, shall be designated by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively.

A plus or minus sign between vectors means a vector sum or difference. For example,

$$\mathbf{E} = \mathbf{E}_x + \mathbf{E}_y + \mathbf{E}_z = E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}$$

means that $\mathbf{E}$ is the vector sum of its components; that is, $\mathbf{E}$ is in magnitude and direction the diagonal of the rectangular parallelopiped with $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ as adjacent edges. The magnitude of $\mathbf{E}$ is seen to be given by the scalar equation

$$E^2 = E_x^2 + E_y^2 + E_z^2,$$

in which the plus sign indicates ordinary addition.

2. No Simple Method of Computing E.—In the most general case in which there are various conductors and insulators aggregated into a system there is no simple method of computing

the electric intensity **E**. We shall be able to arrive at the laws governing such a system only by successive generalizations from simpler systems. The generalizations made will involve the introduction from time to time of new assumptions which may not have been submitted to immediate experimental tests. Instead of resting on direct tests of the assumptions themselves, the validity of the assumptions may require to be established by tests made on the consequences of the assumptions.

3. Electrical Intensity Due to a Single Point Charge in an Infinite Vacuum.—In this simple case where there is a single point charge in an infinite vacuum the electric intensity at any point distant r from the charge has the magnitude

$$E = q/r^2 \tag{1}$$

The direction of this intensity is the direction of $\mathbf{r}$, so that the magnitude and direction of $\mathbf{E}$ is expressible in the vector equation

$$\mathbf{E} = \frac{q}{r^2}\mathbf{U}_r \tag{2}$$

In these equations

$\mathbf{E}$ = electric intensity at P in dynes per electrostatic unit charge,
q = electric charge at 0 in electrostatic units,
r = distance from 0 to P in centimeters,
$\mathbf{U}_r$ = a unit-vector in direction of r from 0 to P.

The inverse-square law[1] for electric intensity, as expressed in equations (1) and (2), has been put into an integrated form and submitted to rigid experimental tests by Cavendish.[2]

4. Effect of Dielectric on Electric Intensity.—If into the field surrounding the point charge various dielectrics are introduced, the intensity is in general changed in a very complicated way. These various dielectrics are said to have different values of *inductivity*, or *dielectric constant.*[3]

The inductivity, or dielectric constant, of the medium at any point will be designated by ϵ, which is in general a function of the coordinates x, y, z, and in some media (those of a crystalline character) the inductivity is also different in different directions.

If the medium is infinite in extent and is everywhere of the same

[1] Due to Coulomb.

[2] Left in manuscript published by Maxwell in 1879.

[3] Attributed by Faraday to a "certain polarized state of the particles;" *Experimental Researches*, 1295, 1298, and 1304 (1837).

inductivity, the electric intensity is inversely proportional to the inductivity of the medium, and the law of force is given correctly by the equation

$$\mathbf{E} = \frac{q}{\epsilon r^2}\mathbf{U}_r \tag{3}$$

with magnitude

$$E = \frac{q}{\epsilon r^2} \tag{4}$$

where q = *intrinsic* charge (defined in next section).

This proposition is proved by the fact that it gives the proper value for the capacity of a condenser with homogeneous dielectric.

5. Definition of Intrinsic Charge.—In the statement of the law of force immediately preceding, the charge q is designated as *intrinsic charge*. An *Intrinsic Charge* is a charge whose time derivative within a region gives the ordinary electric current flowing into the region. A body which contains an intrinsic charge will suffer a translation if placed unsupported in a uniform electric field. Intrinsic charges are to be distinguished from the induced charges, that are sometimes supposed to exist in dielectrics, in the form of a union of positive and negative charges capable of being oriented under the action of a uniform field, but undergoing no translation in such a field.

In modern electron theory, intrinsic charges are supposed to be due to *free electrons;* and induced charges due to *bound electrons*. The motions of the free electrons throughout conductors constitute the ordinary conduction currents of electricity. This subject will be considered later, but for the present the only charges referred to shall be the intrinsic charges.

6. Electric Induction.—Related to electric intensity it is convenient to employ a second vector called *Electric Induction*, which we shall designate by $\mathbf{D}$, with components $\mathbf{D}_x$, $\mathbf{D}_y$, and $\mathbf{D}_z$. Whether the medium is homogeneous or not the Electric Induction at any point is defined as the product of the electric intensity at the point by the inductivity ϵ of the medium at the point. In a non-crystalline, or isotropic, medium the dielectric constant is the same in all directions, and

$$\left.\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} \\ \mathbf{D}_x &= \epsilon\mathbf{E}_x \\ \mathbf{D}_y &= \epsilon\mathbf{E}_y \\ \mathbf{D}_z &= \epsilon\mathbf{E}_z \end{aligned}\right\} \tag{5}$$

On the other hand, if the medium is crystalline (anisotropic) the dielectric constant at a given point has different values in different directions, and, in general,

$$\left.\begin{aligned} D_x &= \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z \\ D_y &= \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z \\ D_z &= \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z \end{aligned}\right\} \tag{6}$$

7. Definition of Flux of Induction.—At any point P in a given field of force the electric induction has magnitude and direction that are functions of the coordinates of P. Suppose an element of surface dS to be drawn at P, and let the normal to dS have the direction **N**, Fig. 1. If the induction at P is **D**, *the flux of induction through dS is defined as the product of dS by the normal component of* **D**; that is,

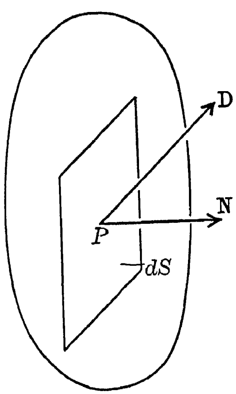

FIG. 1.

$$d\phi_D = DdS \cos (\mathbf{D}, \mathbf{N}) \tag{7}$$

where

$d\phi_D$ = the flux of induction through dS,

D = magnitude of **D**,

$\cos (\mathbf{D}, \mathbf{N})$ = cosine of the angle between **D** and **N**.

The flux of induction through any extended surface S is obtained by integrating $d\phi_D$ over the entire surface:

$$\phi = \int DdS \cos (\mathbf{D}, \mathbf{N}) \tag{8}$$

8. Proof of Gauss's Theorem for a Homogeneous Dielectric. We come now to an important proposition due to Gauss, concerning the flux of induction through a closed surface. Let us suppose that we have throughout a certain region a homogeneous dielectric of dielectric constant ϵ and that there is an intrinsic charge q of electricity concentrated at a point within the region, and let us draw within the homogeneous region any closed surface S completely enclosing the charge q, Fig. 2. At any point P on the surface the electric induction is in the direction of **r** and has, by equations (4) and (5), the magnitude

$$D = q/r^2 \tag{9}$$

The total flux of induction outward through the closed surface is

$$\phi_D = \int D dS \cos \Theta \tag{10}$$

where Θ is the angle between **D** and **N**.

Now if $d\Omega$ is the solid angle subtended at q by dS, it is seen by the geometry of the figure that

$$dS \cos \Theta = r^2 d\Omega \tag{11}$$

whence, by substitution of (9) and (11) in (10),

$$\phi_D = q \int d\Omega = 4\pi q \tag{12}$$

where

ϕ_D = flux of induction outward through the closed surface.

It thus appears that in a homogeneous medium the flux of induction outward through any closed surface is independent of the position of q within the enclosure. The limitation that q is to be concentrated at a point may hence be removed, and *the charge q may be distributed in any manner whatever within the enclosure.*

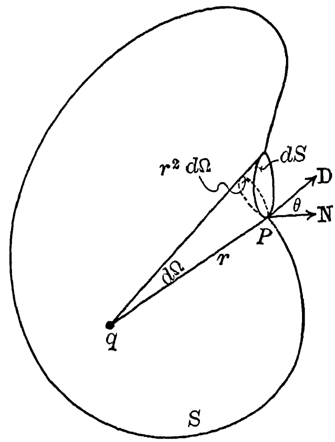

FIG 2

If on the other hand we have a charge q_0 within the homogeneous medium but outside of the enclosure, Fig. 3, and if we draw a solid angle $d\Omega$ at q_0, intercepting, from the closed surface, elements dS_1, dS_2, etc., it will be seen that at every element dS_1 where the direction of **r** is into the enclosure, cos (**r**, **N**) is negative; therefore,

$$\frac{dS_1 \cos (\mathbf{D}, \mathbf{N})}{r^2} = -\, d\Omega;$$

and at every element dS_2 at which **r** points out from the enclosure, cos (**r**, **N**) is positive; therefore,

$$\frac{dS_2 \cos (\mathbf{D}, \mathbf{N})}{r^2} = +\, d\Omega;$$

and that there are as many positive elements as negative elements; hence the flux of induction outward through all the ele-

ments intercepted by $d\Omega$ is zero. Therefore, *the total flux of induction through a closed surface due to a charge outside of the enclosure is zero.*

For charges both inside and out, the result may be summed up as follows:

Gauss's Theorem.—The total flux of electric induction outward through any closed surface due to charges partly within the enclosure and partly outside of it is 4π times the quantity of intrinsic electricity within the enclosure.

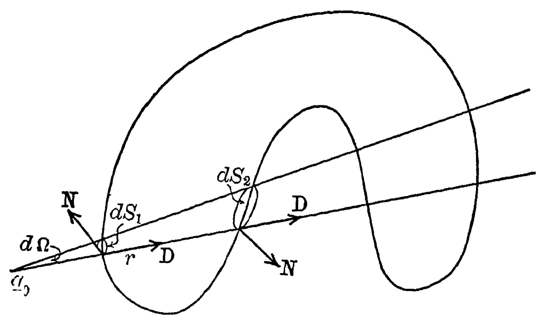

FIG. 3.

9. Limitation Under Which Gauss's Theorem has been Deduced.—In the preceding section we have started with a very limited experimental result that the electric intensity due to a point charge *in a uniform medium* is that given by equation (4). To this we have added the definition of induction given in equation (5). From this limited material we have deduced Gauss's equation

$$\phi_D = 4\pi q$$

which is rigorously established for a uniform medium

The derived result is less definitive of **D** than the original equation (4). This is evident from the consideration that with a given distribution of intrinsic charges the elementary equation (4) would determine one and only one value of the induction $\mathbf{D}_1$ (say) at a given point; whereas Gauss's equation would be satisfied by $\mathbf{D}_1$ plus any other vector $\mathbf{D}_0$ such that the surface integral of $\mathbf{D}_0$ over the closed surface is zero.

10. Assumption that Gauss's Theorem is Perfectly General. Equation (12), Gauss's Theorem, is in accord with the equation (4) and the definition (5) when the dielectric is uniform, and is

therefore in accord with experiments performed on uniform dielectrics; for example, experiments on the capacity of condensers.

As the next step in our search for general laws of the electric field, we are going to assume that Gauss's Theorem without any modification whatever is perfectly general for every possible distribution of charges, conductors, and dielectrics at rest. The justification of this assumption is to be sought in a comparison of experimental results with deductions from the assumption.

11. Gauss's Theorem Expressed in Terms of a Point Relation. We shall next express Gauss's Theorem in terms of a point-relation. Let us take a point whose coordinates are x, y, and z, and for our closed surface, let us take the surface of the elemental volume

$$\Delta\tau = \Delta x \Delta y \Delta z \tag{13}$$

Let ρ be the intrinsic density of electricity at the point x, y, z, and let ρ be the average density in the elemental volume; then the total intrinsic quantity of electricity in the volume is

$$\begin{aligned} \Delta q &= \bar{\rho}\Delta x \Delta y \Delta z \\ &= \bar{\rho}\Delta\tau \end{aligned} \tag{14}$$

whence by Gauss's Theorem, equation (12), the flux of induction is

$$\Delta\phi_D = 4\pi\bar{\rho}\Delta\tau \tag{15}$$

or taking the limit as $\Delta\tau$ approaches zero

$$\frac{\partial\phi_D}{\partial\tau} = 4\pi\rho \tag{16}$$

The left-hand side of this equation is seen to be the limit as the volume approaches zero of the flux outward of **D** from a small volume divided by the volume. This quantity is called the *divergence* of **D**. There follows a digression in which the divergence of a vector is obtained in a different form.

12. Digression on the Divergence of a Vector.—Let ϕ_A be the surface integral of the outward normal component of any vector **A** over a closed surface, and let it be required to find an analytical expression for the limit of the ratio of the surface integral to the volume as this volume approaches zero.

In Fig. 4 is represented the element of volume $\Delta x\Delta y\Delta z$ with one of its corners at the point x, y, z. Let **A** be a vector whose components are analytic functions of the coördinates x, y, z. Let $\bar{\mathbf{A}}_x$ be the average value of the x-component of **A** over the

surface (1). This quantity is in the direction of the normal inward to the surface. The average value over the opposite surface (2) is, by Taylor's Theorem, $\bar{\mathbf{A}}_x + \frac{\partial \bar{\mathbf{A}}_x}{\partial x}\Delta x + \quad .$, and is seen to be outward.

Likewise the average normal component of $\mathbf{A}$ at the surface (3) is $\bar{\mathbf{A}}_y$ inward, and that at the surface (4) is $\bar{\mathbf{A}}_y + \frac{\partial \bar{\mathbf{A}}_y}{\partial y}\Delta y + \ . \ . \ .$ outward. Similarly for the other two faces of the element, which are perpendicular to the z-axis, the average normal components of the vector are respectively $\bar{\mathbf{A}}_z$ inward and $\bar{\mathbf{A}}_z + \frac{\partial \bar{\mathbf{A}}_z}{\partial z}\Delta z + \ . \ . \ .$ outward.

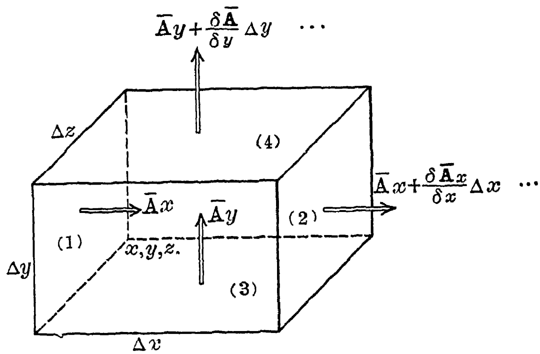

FIG. 4.

Giving a minus sign to the normal vectors that are inward, and multiplying the magnitude of each of the normal terms by the corresponding area of the face of the element through which it acts, we have, as the total outward normal surface integral, the equation

$$
\left.
\begin{aligned}
\int A_n dS = \phi_A = -\bar{A}_x\Delta y\Delta z + &\left\{\bar{A}_x + \frac{\partial \bar{A}_x}{\partial x}\Delta x + \ . \ . \ . \right\}\Delta y\Delta z \\
-\bar{A}_y\Delta x\Delta z + &\left\{\bar{A}_y + \frac{\partial \bar{A}_y}{\partial y}\Delta y + \ . \ . \ \right\}\Delta x\Delta z \\
-\bar{A}_z\Delta x\Delta y + &\left\{\bar{A}_z + \frac{\partial \bar{A}_z}{\partial z}\Delta z + \ . \ . \ . \right\}\Delta x\Delta y
\end{aligned}
\right\} \quad (17)
$$

Dividing by $\Delta x\Delta y\Delta z = \Delta\tau$ and taking the limit as $\Delta\tau$ approaches zero we have

$$
\underset{\Delta\tau = 0}{\text{Lim.}}\left[\frac{\int A_n dS}{\Delta\tau}\right] = \frac{\partial \phi_A}{\partial \tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (18)
$$

where the derivative with respect to τ is a partial derivative because ϕ_A may be regarded as a function of x, y, z and τ; so that the partial derivative with respect to τ means the derivative at a fixed point x, y, z.

Equation (18) may be briefly written

$$\operatorname*{Lim.}_{\Delta\tau = 0}\left[\frac{\int A_n dS}{\Delta\tau}\right] = \frac{\partial \phi_A}{\partial \tau} = \text{div. } \mathbf{A} \tag{19}$$

where

$$\text{div. } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{20}$$

The divergence of a vector **A** *is the flux of the vector outward from a small volume divided by the volume. It is a scalar quantity, has in general different values at different points, and may be obtained direcly by performing the operation indicated in equation* (20).[1]

13. Poisson's Equation.—In view of equation (20) we may now express equation (16) as follows:

$$\text{div. } \mathbf{D} = 4\pi\rho \tag{21}$$

wherever ρ is finite.

The divergence of electrical induction at any point where ρ is finite is 4π times the intrinsic charge density ρ at the point. Equation (21) is known as Poisson's equation.

At all points in space where there is zero intrinsic charge density

$$\text{div. } \mathbf{D} = 0 \tag{22}$$

14. Gauss's Theorem Applied to a Surface Distribution. Surface Divergence.—Suppose that there is an intrinsic charge distributed over a surface, with a surface density σ. At a point in such a surface ρ is no longer finite, so that Gauss's Theorem cannot be reduced to the divergence equation (21), but is preferably reduced to a new point relation as follows:

[1] Assumptions have been made in sections **10** and **11** as follows:

1. In passing to the limit in deriving (16) it was assumed that the intrinsic charge density ρ at the point x, y, z, is spatially continuous in such a way that for a sufficiently small region about x, y, z the average density differs from the density at the point by an amount less than any predetermined quantity.

2 It was assumed that **A** is a function of x, y, z of a form capable of being developed by Taylor's theorem.

At any required point on the surface (Fig. 5) let us mark out an element of surface ΔS, and through the periphery of ΔS, draw lines in the direction of the electric induction. These lines bound a short tube of induction, which we shall suppose to be terminated by the surface elements ΔS_1 and ΔS_2 parallel to ΔS. Let h be the distance between ΔS_1 and ΔS_2. Over the convex surface of the tube the normal component of induction is everywhere zero, since the induction is in the direction of the convex surface. Over the ends of the tube, let the average component of induction away from ΔS be $\overline{D}_{1n1}$ and $\overline{D}_{2n2}$. Then by Gauss's Theorem

$$\overline{D}_{1n1}\Delta S_1 + \overline{D}_{2n2}\Delta S_2 = 4\pi\bar{\sigma}\Delta S \tag{23}$$

If now we allow h, the height of the tube, to approach zero, ΔS_1 and ΔS_2 both approach ΔS as a limit; whence

$$\overline{D}_{1n1} + \overline{D}_{2n2} = 4\pi\bar{\sigma} \tag{24}$$

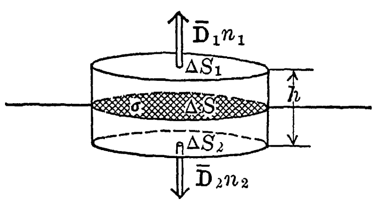

FIG 5.

If now we allow the surface ΔS to shrink toward a point P on the charged surface, the average values in (24) may be replaced by their true values at the point, giving

$$D_{1n1} + D_{2n2} = 4\pi\sigma \tag{25}$$

in which D_{1n1} and D_{2n2} are both drawn away from the charged surface. The sum of the two normal components thus drawn is called by Abraham and Foppl[1] the *surface divergence* of the vector **D**. The result (25) may be stated as follows:

The surface divergence of induction at any point is 4π times the intrinsic surface charge density at the point.

As a corollary, the surface divergence of induction is zero at all points where there is no intrinsic surface charge.

If instead of drawing the two normals both away from the surface under consideration, one of them be reversed so that they

[1]Abraham und Föppl. Theorie der Elektricität, Vol. 1, p. 77, 1907.

point in the same sense through the surface, equation (25) becomes

$$\left.\begin{aligned} D_{1n1} - D_{2n1} &= 4\pi\sigma \\ \text{or} \qquad & \\ D_{2n2} - D_{1n2} &= 4\pi\sigma \end{aligned}\right\} \tag{26}$$

that is, *there is a discontinuity in the magnitude of the normal component of* **D** *amounting to* $4\pi\sigma$, *where* σ *is the intrinsic surface density.*

15. Analogous Treatment of Magnetic Field.—In a field of magnetic force, the force at any point per unit magnetic pole is called the *Magnetic Intensity* and is designated by **H.** The unit magnetic pole is a pole that will repel an equal pole at a distance of one centimeter with a force of one dyne in vacuo. The product of the magnetic intensity by the permeability of the medium at the point is called *Magnetic Induction,* and is designated by **B.**

$$\left.\begin{aligned} \mathbf{B} &= \mu\mathbf{H} \\ \mathbf{B}_x &= \mu\mathbf{H}_x \\ \mathbf{B}_y &= \mu\mathbf{H}_y \\ \mathbf{B}_z &= \mu\mathbf{H}_z \end{aligned}\right\} \tag{27}$$

where

$$\mu = \text{magnetic permeability.}$$

The question whether there is or is not any intrinsic volume density of magnetism is open to disputation. It is proposed to limit the discussion in the present work to cases where this volume density is zero; so that reasoning similar to that used in the discussion of electrical quantities in the preceding paragraphs gives from the inverse square law for a uniform magnetic medium the result

$$\text{div.}\,\mathbf{B} = 0 \tag{28}$$

and this is assumed to be universally true.

Also in all cases that will come under our observation

$$\text{surf. div.}\,\mathbf{B} = 0 \tag{29}$$

CHAPTER II

MAXWELL'S EQUATIONS

16. Summary of Chapter I.—The important results obtained in the preceding chapter are contained in the following equations, which are taken with their original numerical designations·

$$\text{div. } \mathbf{D} = 4\pi\rho, \text{ wherever } \rho \text{ is finite,} \qquad (21), \text{ Ch. I.}$$
$$\text{surf. div. } \mathbf{D} = 4\pi\sigma, \qquad (25), \text{ Ch. I.}$$
$$\text{div. } \mathbf{B} = 0, \qquad (28), \text{ Ch. I.}$$
$$\text{surf. div. } \mathbf{B} = 0, \qquad (29), \text{ Ch. I.}$$

where **D** and **B** are respectively electric and magnetic induction at any point, ρ is intrinsic volume density of electric charge, and σ is intrinsic surface density of electric charge at the point.

The electric intensity **E** can be obtained by dividing **D** by the dielectric constant ϵ; the magnetic intensity **H** can be obtained by dividing **B** by the permeability μ.

The above equations are not sufficient to determine **D** and **B**.

17. Note as to Additional Requirements.—In addition to the divergence of a vector we need also its *curl*, which is a related vector to be later defined. These two quantities, divergence and curl, together with certain boundary conditions, are sufficient to determine a required vector.

In electrostatics, where there are assumed to be no electric currents or motions of electric charges and no variations of **D** and **B** with the time, it can be shown that the curl of **D** and the curl of **B** are both zero. It can then be shown that a scalar potential function exists, and familiar methods are at hand for completely determining **D**, **B**, **E**, and **H** in cases where proper boundary conditions are given.

When, however, we leave the field of electrostatics and enter upon the general problem, the curls of **D** and **B** are no longer zero, the scalar potential functions for these vectors have no existence, and the older theoretical investigations of Laplace and of Poisson are insufficient to describe the characteristics of the electromagnetic field.

The way to proceed under these more difficult conditions was pointed out by Maxwell in 1865–6, in a mathematical research which contained a prediction of the existence of electric waves, determined the velocity of propagation of the waves, and explained the nature of light.

18. Further Experimental Relations for the Electromagnetic Field.—In developing the theory of electric waves, we may make use of the following experimental laws:

I. THE M.M.F. EQUATION.—The work done by the magnetic field in carrying a unit magnetic pole once around a closed path, Fig. 1, linking positively with a *closed* circuit carrying a steady current I is

$$W = 4\pi I \tag{1}$$

in which W is work in ergs per unit pole, and I is current in absolute c.g.s. electromagnetic units of current (absamperes).

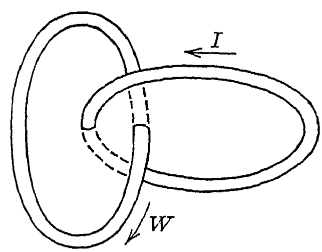

FIG. 1—Arrows marked W and I point in the direction of positive linkage.

Throughout this volume, in order to obtain symmetrical results, we shall measure all *electrical* quantities in absolute c.g.s. *electrostatic* units, and all *magnetic* quantities in absolute c.g.s. *electromagnetic* units. Such a composite system of units is called the *Gaussian* system. In Gaussian units, equation (1) becomes

$$W = \frac{4\pi I}{c} \tag{2}$$

in which

c = the number of electrostatic units of quantity of electricity in one electromagnetic unit.[1]

II. THE E.M.F. EQUATION—The electromotive force produced in a closed circuit, Fig. 2, by varying the flux of magnetic induction linking with it positively is

$$V = -\frac{\partial \phi_B}{\partial t} \tag{3}$$

in which V is in electromagnetic units (abvolts). If we put V

[1] It is a characteristic of the Gaussian units that c always enters along with t, whether t is expressed as in (4) or implied as in (2)—implied in that the current I is quantity per unit time.

into electrostatic units so as to conform with the Gaussian system as above specified, we have

$$V = -\frac{1}{c}\frac{\partial \phi_B}{\partial t} \tag{4}$$

The direction of positive linkage is shown in the diagrams of Figs. 1 and 2.

Equations (2) and (4) are called respectively the magnetomotive force equation and the electromotive force equation, abbreviated M.M.F. and E.M.F.

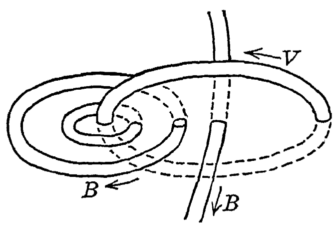

FIG. 2.—Arrows marked V and B point in direction of positive linkage.

They are now to be transformed into point relations. For this purpose a system of rectangular axes is chosen as follows.

19. Choice of Axes.—Following what now seems to be the prevalent usage in electromagnetic theory, we shall adopt as our system of rectangular axes the system shown in Fig. 3, in which z points out from the plane of the paper toward the reader, when x is to the right and y is upward in the plane of the paper. This rule merely gives the relative orientation of the axes, and it is evident that the scheme of Fig. 4 is the same system of axes.

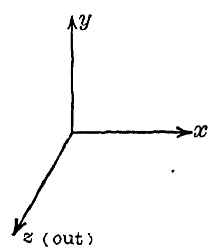

FIG. 3.—Positive set of axes

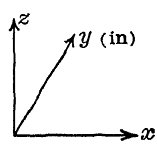

FIG. 4.—Also positive set of axes

20. Transformation of Magnetomotive Force Equation into a Point Relation.—Let us take any extended region (for example, the room of a building) and suppose that there are electric currents flowing in conducting masses within the room, and let the current density at any point x, y, z be $\mathbf{u}$ with components $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$, along the three axes respectively. As a special case $\mathbf{u}$ may be zero at some or all points.

Let us now consider the magnetomotive force around a rectangle $\Delta y \Delta z$, Fig. 5, drawn with one corner at the point x, y, z. The component of current density at the point x, y, z perpendicular to the area $\Delta y \Delta z$ is $\mathbf{u}_x$. The other components of current density, those in the directions y and z, contribute nothing to the M.M.F. around the area.

Now the average value of $\mathbf{u}_x$ over the area is different from $\mathbf{u}_x$ at x, y, z, and we shall designate this average value by $\bar{\mathbf{u}}_x$.

The current through the area is then

$$I = \bar{u}_x \Delta y \Delta z \tag{5}$$

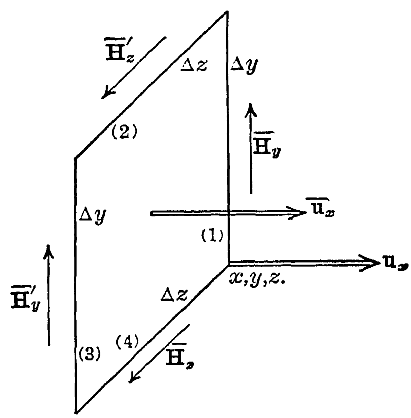

FIG. 5.

whence by equation (2) the M.M.F. around the area is

$$W = \frac{4\pi \bar{u}_x}{c} \Delta y \Delta z \tag{6}$$

Let us now get a second expression for this M.M.F., W, by estimating directly from the geometry of the problem the work done by the magnetic forces in driving a unit magnetic pole around the area $\Delta y \Delta z$. The magnetic force on a unit pole at the point x, y, z is $\mathbf{H}$, with components $\mathbf{H}_x$, $\mathbf{H}_y$, and $\mathbf{H}_z$ along the three axes. The magnetic force and its components are different for different points of the region.

Since the work by a force in displacing its point of application is the magnitude of the force times the displacement in the direction of the force, we shall have for the work of carrying a

unit magnetic pole in the positive direction around the rectangle the equation

$$W = \overline{H}_y\Delta y + \overline{H}'_z\Delta z - \overline{H}'_y\Delta y - \overline{H}_z\Delta z \qquad (7)$$

in which $\overline{H}_y$ is the average value of H_y along the side (1), $\overline{H}'_z$ is the average value of H_z along the side (2), etc.

Now $\overline{H}_y$ is a function of x, y, z, and Δy, and may be written

$$\overline{H}_y = H_y(x, y, z, \Delta y).$$

Also we may write

$$\overline{H}'_y = H_y(x, y, z + \Delta z, \Delta y)$$

which is the same function with $z + \Delta z$ substituted for z;

whence by Taylor's Theorem, assuming proper continuity and writing only first order terms,

$$\overline{H}'_y = \overline{H}_y + \frac{\partial \overline{H}_y}{\partial z}\Delta z \qquad (8)$$

In like manner,

$$\overline{H}'_z = \overline{H}_z + \frac{\partial \overline{H}_z}{\partial y}\Delta y \qquad (9)$$

Taking the right-hand side of (6) and equating it to the right-hand side of (7) after replacing $\overline{H}'_y$ and $\overline{H}'_z$ by their values from (8) and (9), we have

$$\frac{4\pi u_x}{c}\Delta y\Delta z = \left(\frac{\partial \overline{H}_z}{\partial y} - \frac{\partial \overline{H}_y}{\partial z}\right)\Delta y\Delta z \qquad (10)$$

Dividing through by $\Delta y\Delta z$ and taking the limit as Δy and Δz approach zero, we have

$$\frac{4\pi u_x}{c} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z},$$

and by similar reasoning,

$$\left.\begin{aligned}\frac{4\pi u_y}{c} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\\ \frac{4\pi u_z}{c} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\end{aligned}\right\} \qquad (11)$$

or briefly the vector equation

$$\frac{4\pi \mathbf{u}}{c} = \text{curl } \mathbf{H} \qquad (12)$$

where curl H is a vector, the magnitude of whose x, y, and z components[1] are respectively

$$\left.\begin{aligned} \text{curl}_x\,\mathbf{H} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \text{curl}_y\,\mathbf{H} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \text{curl}_z\,\mathbf{H} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \tag{13}$$

and where

$$\text{curl}\,\mathbf{H} = \mathbf{i}\,\text{curl}_x\,\mathbf{H} + \mathbf{j}\,\text{curl}_y\,\mathbf{H} + \mathbf{k}\,\text{curl}_z\,\mathbf{H} \tag{14}$$

The vector equation (12) *or the equivalent Cartesian equations* (11) *give a relation*[2] *between the electric current density at a point (in electrostatic units) and the magnetic intensity at the same point (in electromagnetic units) derived under the limitations:*

I. That the vectors **u** and **H** are continuous functions of the coordinates of the point; and

II. That the current is of such a character that the original M.M.F. equation (2) is true.

21. Transformation of the Electromotive Force Equation into a Point Relation.—We shall now transform the other fundamental equation (4) into a form analogous to (12), and obtain a second set of Maxwell's equations. We can do this by the similarity of the equations (2) and (4), without going again through the details of a demonstration like the preceding.

It is to be noted that W of (2) is a line integral of the magnetic intensity **H** around a closed curve. Likewise, in (4) the electromotive force V, defined as the work by the field in driving a unit charge around a closed circuit, is a line integral of the electric intensity **E** around the circuit. Also the magnetic induction **B** is related to the flux of induction ϕ_B in the same way that current

[1] From the above analysis it is seen that the method of obtaining the component of curl **H** in any particular direction **N** at any particular point P is as follows:

Surround P by a closed curve S in a plane perpendicular to **N**. Let ΔS be the area within the curve, then

$$\text{curl}_N\,\mathbf{H} = \underset{\Delta S \doteq 0}{\text{limit}} \left[\frac{\int H ds \cos(\mathbf{H}, \mathbf{S})}{\Delta S}\right];$$

that is, *curl*$_N$**H** *is the line integral of* **H** *around the periphery of a small area perpendicular to* **N** *divided by the small area.*

[2] Maxwell: "Electricity and Magnetism," Vol. II.

density $\mathbf{u}$ is related to current I. This shows that by going through a process similar to that employed in transforming (2), we should obtain from (4) the equations

$$\left.\begin{aligned} -\frac{1}{c}\frac{\partial B_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ -\frac{1}{c}\frac{\partial B_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ -\frac{1}{c}\frac{\partial B_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{aligned}\right\} \tag{15}$$

or, briefly, in vectorial notation,

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \text{curl } \mathbf{E} \tag{16}$$

The vector equation (16), *or the equivalent Cartesian equations* (15), *gives a relation*[1] *between the time derivative of the magnetic induction at a point and the electric intensity at the same point derived under the limitations:*

I. That the vectors $\mathbf{B}$ and $\mathbf{E}$ are continuous functions of x, y, z, and t; and

II. That the original electromotive force equation (4) is a correct experimental law.

Equations (15) or (16) may be called *Maxwell's Magnetic Induction Equations.*

22. Further Examination of the Two Curl-equations.—In the preceding sections we have derived the equations

$$\frac{4\pi\mathbf{u}}{c} = \text{curl } \mathbf{H} \qquad (12) \ \S\mathbf{20}$$

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \text{curl } \mathbf{E} \qquad (16) \ \S\mathbf{21}$$

which may be called respectively the "current-density equation" and the "magnetic induction equation."

These equations were derived subject to the assumption that the M.M.F. law (2) *and the E.M.F. law* (4) *are correct and general.*

We shall now show that the current-density equation (12) of §**20** cannot be true in general; for the reason that the divergence of any curl (*e.g.*, div. curl $\mathbf{A}$) is zero, while the divergence of $\mathbf{u}$ is not zero except in a special case in which the quantity of electricity flowing out of a given region in a given time is equal to the

[1] Maxwell: "Electricity and Magnetism," Vol. II.

quantity flowing in. The separate steps of this demonstration will now be given.

23. Theorem. Div. Curl A = 0.—Proof:

$$\begin{aligned}\text{div. curl } \mathbf{A} &= \frac{\partial}{\partial x}(\text{curl}_x \mathbf{A}) + \frac{\partial}{\partial y}(\text{curl}_y \mathbf{A}) + \frac{\partial}{\partial z}(\text{curl}_z \mathbf{A}) \\ &= \frac{\partial}{\partial x}\left\{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right\} + \\ &\qquad \frac{\partial}{\partial z}\left\{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right\} \\ &= 0 \end{aligned} \tag{17}$$

The last step is conditioned on the equality of such quantities as

$$\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x}\right)$$

These quantities are equal provided it is permissible to change the order of differentiation, and this is permissible provided the second order derivatives so obtained all exist. The conclusion then is that the divergence of the curl of any vector **A** is zero, provided **A** is of such a character that the several second order derivatives of each of its components all exist.

24. Application of This Theorem to (12) and (16).—Taking the divergence of both sides of the current-density equation (12) and the magnetic-induction equation (16), we have, respectively

$$\text{div. } \mathbf{u} = 0 \tag{18}$$

$$\frac{\partial}{\partial t}\left(\text{div. } \mathbf{B}\right) = 0 \tag{19}$$

Now (19) is true, for by §**16** div. **B** = 0. There is, hence, no inconsistency in the magnetic-induction equation (16).

On the other hand, we shall now show that (18) div. **u** = 0 is sometimes true and sometimes not true; to wit, div. **u** = 0 is true when and where there is no changing intrinsic charge density; but div. **u** does not equal zero when and where there is a changing intrinsic charge density. Let us proceed to a critical examination of div. **u**.

25. Examination of Div. u.—If we take any small volume $\Delta\tau$ surrounding a given point P, the quantity of electricity per second flowing out of $\Delta\tau$ is the surface integral of the outward normal component of **u** over the closed surface bounding $\Delta\tau$. This

quantity flowing out is also the time rate of decrease of the quantity of electricity within $\Delta\tau$. Equating these two expressions we have

$$\int u_n dS = -\frac{\partial}{\partial t}\left(\bar{\rho}\,\Delta\tau\right)$$

Dividing by $\Delta\tau$ and taking the limit as $\Delta\tau$ approaches zero, we have

$$\underset{\Delta\tau \doteq 0}{\text{Limit}}\left[\frac{\int u_n dS}{\Delta\tau}\right] = -\frac{\partial\rho}{\partial t}$$

but by definition of divergence, equation (19), Art. **12,** the left-hand side is the divergence of **u**, hence

$$\text{div.}\,\mathbf{u} = -\frac{\partial\rho}{\partial t} \tag{20}$$

A similar treatment shows that at any point in a charged surface

$$\text{surf. div.}\,\mathbf{u} = -\frac{\partial\sigma}{\partial t} \tag{21}$$

These equations (20) and (21) express the fact that the quantity of electricity flowing out from a small region in a given time is equal to the decrease of the quantity of intrinsic electricity within the region; that is, these equations are a statement of the law of the conservation of electricity.

We have thus a proof of the lack of generality of the current-density equation (12), and hence a lack of generality in the original M.M.F. equation (2).

The conclusion is that the original equation

$$W = \frac{4\pi I}{c},$$

and the derived equation

$$\frac{4\pi\mathbf{u}}{c} = \text{curl}\,\mathbf{H}$$

can be true only when the electric current is such that $\dfrac{\partial\rho}{\partial t} = 0$; *that is, such that there is no fluctuating accumulation of electricity.*

26. Condensive and Non-condensive Flow.—If we have a conductor of electricity in Fig. 6, with an electric current I flowing in it, and if P is any point on or within the conductor, and if we enclose a small region around P, and if I is varying with the time, the small region around P will have a small electro-

static capacity, and will in general variously charge and discharge with the time. We may call such a flow of electricity a *Condensive Flow*, since there is an action similar to that of a condenser at P. If on the other hand, the current is in a steady state, there is no such fluctuation of charge at P, and the amount of electricity flowing out of the small region is at any time equal to the amount of electricity flowing in. This may be called a *Non-condensive Flow*

$$\text{For a non-condensive flow div. } \mathbf{u} = 0;$$
$$\text{For a condensive flow div. } \mathbf{u} \neq 0.$$

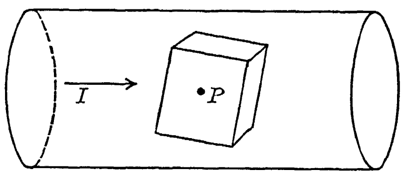

FIG. 6—Illustrating condensive and non-condensive flow.

27. Maxwell's Displacement Assumption.—We have shown that the equation

$$\frac{4\pi\mathbf{u}}{c} = \text{curl } \mathbf{H}$$

can be true only in those cases where

$$\text{div. } \mathbf{u} = 0;$$

that is, in a non-condensive flow. In such a non-condensive flow the current density $\mathbf{u}$ at a point P may be said to have associated with it a magnetic field of intensity $\mathbf{H}$ at the point, and the relation of $\mathbf{u}$ to $\mathbf{H}$ is that given above.

On the other hand, in the general case where the flow may be condensive or non-condensive, we must replace u, the ordinary intrinsic density, by some other quantity $\mathbf{u}'$, such that

$$\text{div. } \mathbf{u}' = 0.$$

A vector whose divergence is zero is called a *solenoidal vector.* Maxwell made the assumption that the appropriate solenoidal vector $\mathbf{u}'$ by which the non-solenoidal vector $\mathbf{u}$ should be replaced is

$$\mathbf{u}' = \mathbf{u} + \frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t} \tag{22}$$

It is very apparent that the quantity here added to $\mathbf{u}$ is just sufficient to make the sum $\mathbf{u}'$ solenoidal; for by (20) and (21),

$$\text{div.}\,\mathbf{u} = -\frac{\partial \rho}{\partial t}, \qquad \text{surf div.}\,\mathbf{u} = -\frac{\partial \sigma}{\partial t},$$

and by Art **13,** equations (21) and (25) (their time derivatives)

$$\text{div.}\left\{\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}\right\} = \frac{\partial \rho}{\partial t}, \qquad \text{surf. div.}\left\{\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}\right\} = \frac{\partial \sigma}{\partial t}.$$

Summing these quantities, we have

$$\text{div.}\,\mathbf{u}' = 0, \qquad \text{surf. div.}\,\mathbf{u}' = 0 \tag{23}$$

Maxwell's Assumption is the assumption that in respect to the Magnetic Field the quantity $\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}$ *acts as a density of current, which he called* **displacement current,** *and which must be added to conduction current density* $\mathbf{u}$ *to give complete current density* $\mathbf{u}'$.

28. The Generalized Current Density Equation.—With this assumption the current density equation (12) may be generalized into

$$\frac{4\pi\mathbf{u}}{c} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \text{curl}\,\mathbf{H} \tag{24}$$

which may be called *Maxwell's Generalized Current-density Equation.* The addition of the first two terms is a vector addition.

It is apparent that there is no mathematical inconsistency in Maxwell's method of generalizing the conception of an electric current, in respect to its effect in producing or responding to a magnetic field. Whether or not this generalized current is related to the magnetic field intensity by an equation of the form of (24) is a question for experimental determination. The experimental test has never been adequately made on the assumption directly. The validity of Maxwell's Assumption rests on his prediction from it of the existence of electric waves, and on his prediction of the electromagnetic character of light. These predictions have been amply verified.

29. At any Surface of Electric or Magnetic Discontinuity the Tangential Components of E and H are Continuous.—We need the proposition here stated, for the solution of problems pertaining to surfaces of discontinuity. It may be proved as follows: Referring to Fig. 7, at any surface of discontinuity in conductivity, dielectric constant or permeability, let us draw a

small elongated rectangle with its length a parallel to the surface of discontinuity, and let b be the width of the rectangle. Let $\overline{E}_{1T}$, $\overline{E}_{2T}$, $\overline{E}_3$, and $\overline{E}_4$ be the average values of the electric intensity along the four sides of the rectangle; and let $\overline{B}$ be the average value of magnetic induction perpendicular to the rectangle; then we have by the E.M.F. equation (3) the result

$$-\frac{1}{c}\frac{\partial}{\partial t}(\overline{B}\,ab) = -\overline{E}_{1T}a + \overline{E}_3 b + \overline{E}_{2T}a - \overline{E}_4 b$$

If now we assume that B and E are everywhere finite, and let b approach zero, the left-hand side of the equation approaches zero; also $\overline{E}_3 b$ and $\overline{E}_4 b$ approach zero; hence

$$[\overline{E}_{1T}a = \overline{E}_{2T}a], \text{ for } b = 0$$

and if we let a also approach zero, the average values of E along the sides a approach the actual values at a point on the surface; whence

$$E_{1T} = E_{2T} \tag{25}$$

Hence *the tangential component of* **E** *is everywhere continuous.*

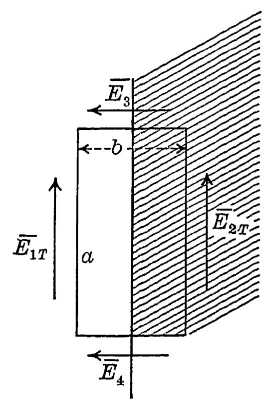

FIG. 7—In proof of continuity of the tangential component of **E**.

In like manner, if the current density **u** is everywhere finite, it can be shown from the M.M.F. equation of the form of (2), with I replaced by a surface integral of **u** and with **u**′ replacing **u** to give the equation generality, that *the tangential component of* **H** *is everywhere continuous.*

CHAPTER III

ENERGY OF THE ELECTROMAGNETIC FIELD. POYNTING'S VECTOR

30. Summary of Chapters I and II.—The important results obtained in the preceding chapters may be summarized in the following equations (in which partial derivative with respect to time is indicated by a dot over a symbol):

$$\frac{4\pi \mathbf{u}}{c} + \frac{\dot{\mathbf{D}}}{c} = \text{curl } \mathbf{H} \tag{A}$$

$$-\frac{\dot{\mathbf{B}}}{c} = \text{curl } \mathbf{E} \tag{B}$$

$$\text{div. } \mathbf{D} = 4\pi\rho \tag{C}$$

$$\text{div. } \mathbf{B} = 0 \tag{D}$$

$$\mathbf{D} = \epsilon\mathbf{E} \tag{E}$$

$$\mathbf{B} = \mu\mathbf{H} \tag{F}$$

To these may be added an expression for the current density $\mathbf{u}$ (derived from Ohm's Law)

$$\left.\begin{aligned} \mathbf{u} &= \gamma\mathbf{E} \\ \text{where} \qquad \gamma &= \text{specific conductivity.} \end{aligned}\right\} \tag{G}$$

We have also derived the following surface relations that hold at surfaces of discontinuity

$$\text{surf. div. } \mathbf{D} = 4\pi\sigma \tag{H}$$

$$\text{surf. div. } \mathbf{B} = 0 \tag{I}$$

$$E_{1T} = E_{2T} \tag{J}$$

$$H_{1T} = H_{2T} \tag{K}$$

These equations will hereafter be designated by the letters ascribed after them respectively, instead of by the accidental numerical designations with which they first appeared.

In the present chapter we shall treat certain general propositions regarding the energy of the field. For this purpose we need at the outset a few theorems in vector analysis.

31. Scalar and Vector Product.—Let **A** and **B** be any two vectors drawn away from a common point, Fig. 1. These vectors may be written

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{1}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} \tag{2}$$

where **i**, **j**, and **k** are unit vectors along the three axes respectively.

If now we introduce the convention that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},$$
$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j},$$

and take the product of **A** and **B** term by term, we obtain

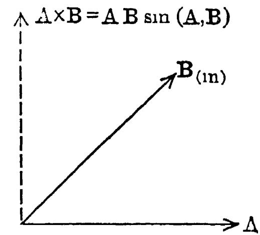

FIG 1.—Illustrating vector product of **A** and **B**.

$$\mathbf{AB} = A_xB_x + A_yB_y + A_zB_z + (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \tag{3}$$

We may call **AB** the *complete product* of **A** by **B**. It is seen to consist of two parts, one of which, consisting of the sum of the first three terms, is scalar; and the other, consisting of the sum of three vector components, is vector. These two parts are called respectively the *scalar product*, to be designated by **A·B** (read "**A** dot **B**"), and the *vector product*, to be designated by **A×B** (read "**A** cross **B**"). Then

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{4}$$

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \tag{5}$$

It is seen that

$$\mathbf{A}\cdot\mathbf{B} = AB\{\cos(\mathbf{A}, x)\cos(\mathbf{B}, x) + \cos(\mathbf{A}, y)\cos(\mathbf{B}, y) + \cos(\mathbf{A}, z)\cos(\mathbf{B}, z)\} \tag{6}$$

$$= AB\cos(\mathbf{A}, \mathbf{B})$$

The scalar product of two vectors is the product of their magnitudes by the cosine of the angle between them.

To find the meaning of the vector product $\mathbf{A}\times\mathbf{B}$, let us designate by l, m, n the direction cosines of $\mathbf{A}$, and by l', m', n' the direction cosines of $\mathbf{B}$. Then the square of the magnitude of $\mathbf{A}\times\mathbf{B}$ is the sum of the squares of the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components; that is

$$[\mathbf{A}\times\mathbf{B}]^2 = \{(mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2\}A^2B^2$$
$$= \{(m^2 + n^2)l'^2 + (n^2 + l^2)m'^2 + (l^2 + m^2)n'^2 - 2(mm'nn' + ll'nn' + mm'll')\}A^2B^2.$$

Now $1 = l^2 + m^2 + n^2 = l'^2 + m'^2 + n'^2$,

whence the preceding equation becomes

$$\begin{aligned}[\mathbf{A}\times\mathbf{B}]^2 &= \mathbf{A}^2\mathbf{B}^2\{1 - (ll' + mm' + nn')^2\}\\ &= A^2B^2\{1 - \cos^2(\mathbf{A}, \mathbf{B})\}\\ &= A^2B^2 \sin^2(\mathbf{A}, \mathbf{B})\end{aligned}$$

therefore,

$$[\mathbf{A}\times\mathbf{B}] = AB \sin (\mathbf{A}, \mathbf{B}) \qquad (7)$$

This gives the magnitude of the vector product. Let us next determine its direction. This can be done by taking the scalar product of $\mathbf{A}$ and $\mathbf{A}\times\mathbf{B}$, which by (4) may be written

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= A_x(A_yB_z - A_zB_y) +\\ &\quad A_y(A_zB_x - A_xB_z) +\\ &\quad A_z(A_xB_y - A_yB_x)\\ &= 0.\end{aligned}$$

In like manner it can be shown that

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0$$

Hence by (6) the vector product $\mathbf{A}\times\mathbf{B}$ is perpendicular to $\mathbf{A}$ and to $\mathbf{B}$. By the convention $\mathbf{ij} = \mathbf{k}$, etc., this perpendicular is to be drawn with respect to $\mathbf{A}$ and $\mathbf{B}$ so that a positive rotation about the product vector will turn $\mathbf{A}$ into the direction of $\mathbf{B}$.

Hence, *the vector product* $\mathbf{A}\times\mathbf{B}$ *is a vector whose magnitude is the product of the magnitudes of* $\mathbf{A}$ *and* $\mathbf{B}$ *by the sine of the angle between them, and whose direction is the positive perpendicular to the plane of* $\mathbf{A}$ *and* $\mathbf{B}$.

The product $\mathbf{B}\times\mathbf{A}$ has the opposite direction, so that

$$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B} \qquad (8)$$

If we make **B** and **A** identical, equations (7) and (6) show that

$$\mathbf{A}\times\mathbf{A} = 0 \tag{9}$$

and

$$\mathbf{A}\cdot\mathbf{A} = A^2 \tag{10}$$

32. The Divergence of a Vector Product.—The divergence of $\mathbf{A}\times\mathbf{B}$ may be found directly as follows:

$$\begin{aligned}
\text{div. }\mathbf{A}\times\mathbf{B} &= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \\
&\quad \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \\
&\quad \frac{\partial}{\partial z}(A_xB_y - A_yB_x) \\
&= A_y\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_y}{\partial x} - A_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial A_z}{\partial x} + \\
&\quad A_z\frac{\partial B_x}{\partial y} + B_x\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y} + \\
&\quad A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - B_x\frac{\partial A_y}{\partial z} \\
&= B_x\,\text{curl}_x\mathbf{A} + B_y\,\text{curl}_y\mathbf{A} + B_z\,\text{curl}_z\mathbf{A} - \\
&\quad A_x\,\text{curl}_x\mathbf{B} - A_y\,\text{curl}_y\mathbf{B} - A_z\,\text{curl}_z\mathbf{B} \\
&= \mathbf{B}\cdot\text{curl }\mathbf{A} - \mathbf{A}\cdot\text{curl }\mathbf{B}
\end{aligned} \tag{11}$$

33. Energy and Radiation.—We shall now treat a very important general proposition with respect to the energy and radiation of energy in the electromagnetic field. Let us take any point x, y, z, Fig. 2, and describe at x, y, z an element of volume

$$\Delta\tau = \Delta x\Delta y\Delta z.$$

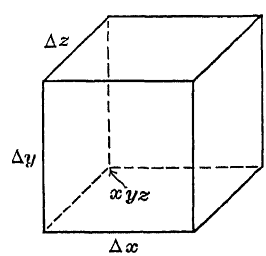

FIG. 2

Suppose that there are current density **u** and electric and magnetic intensities **E** and **H** at x, y, z. Let us study the energy transformations taking place in the volume $\Delta\tau$. The electromotive force between the two opposite $\Delta y\Delta z$-faces of the volume element is the average electric intensity $\overline{\mathbf{E}}_x$ times the distance Δx. The *current* flowing between these faces is the average normal current-density $\mathbf{u}_x$ times the area of one of these faces $\Delta y\Delta z$. Whence the electrical power (energy per second)

converted into heat or other form of power by the current in the x-direction is $\overline{\mathbf{E}}_x\overline{\mathbf{u}}_x\ \Delta x\Delta y\Delta z$. Likewise, the power expended by currents in the y and z-directions is $\overline{\mathbf{E}}_y\overline{\mathbf{u}}_y\Delta x\Delta y\Delta z$ and $\overline{\mathbf{E}}_z\overline{\mathbf{u}}_z\Delta x\Delta y\Delta z$ respectively.

Adding these three quantities we have for the electrical power converted into other forms of power in the element the value

$$\Delta P = (\overline{\mathbf{E}}_x\overline{\mathbf{u}}_x + \overline{\mathbf{E}}_y\overline{\mathbf{u}}_y + \overline{\mathbf{E}}_z\overline{\mathbf{u}}_z)\Delta\tau \tag{12}$$

Dividing by $\Delta\tau$ and taking the limit as $\Delta\tau$ approaches zero, we have, for the power converted per unit volume at x, y, z, the quantity

$$\begin{aligned}\frac{\partial P}{\partial\tau} &= \mathbf{E}_x\mathbf{u}_x + \mathbf{E}_y\mathbf{u}_y + \mathbf{E}_z\mathbf{u}_z\\ &= \mathbf{E}\cdot\mathbf{u}.\end{aligned}$$

Let us now replace **u** by its value from Maxwell's equation (A), obtaining

$$\frac{\partial P}{\partial\tau} = \frac{c}{4\pi}\mathbf{E}\cdot\text{curl }\mathbf{H} - \frac{1}{4\pi}\mathbf{E}\cdot\mathbf{D} \tag{13}$$

Now by the theorem expressed in equation (11)

$$\text{div. }\mathbf{E}\times\mathbf{H} = \mathbf{H}\ \ \text{curl }\mathbf{E} - \mathbf{E}\cdot\text{curl }\mathbf{H}.$$

Substituting the value of **E** curl **H** from this equation into (13), we obtain

$$\frac{\partial P}{\partial\tau} = \frac{c}{4\pi}\ \mathbf{H}\ \ \text{curl }\mathbf{E} - \frac{c}{4\pi}\ \text{div. }\mathbf{E}\times\mathbf{H} - \frac{1}{4\pi}\mathbf{E}\cdot\dot{\mathbf{D}}.$$

Replacing curl **E** by its value from Maxwell's equation (B), we have

$$\frac{\partial P}{\partial\tau} = -\frac{1}{4\pi}\mathbf{H}\cdot\mathbf{B} - \frac{1}{4\pi}\mathbf{E}\cdot\mathbf{D} - \frac{c}{4\pi}\text{div. }\mathbf{E}\times\mathbf{H}.$$

Since ϵ and μ are independent of the time, $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \epsilon\dot{\mathbf{E}}$ and the first two terms may be written as derivatives of squares; and the last term, when multiplied by $\partial\tau$ becomes by (19), Chapter I, a surface integral over the surface of the volume $\partial\tau$; so that

$$\partial P = -\frac{\partial}{\partial t}\left(\frac{\mu}{8\pi}H^2 + \frac{\epsilon}{8\pi}E^2\right)\partial\tau - \frac{c}{4\pi}\int(\mathbf{E}\times\mathbf{H})_n dS \tag{14}$$

In this equation $(\mathbf{E}\times\mathbf{H})_n$ is the outward normal component of the vector $\mathbf{E}\times\mathbf{H}$, and the integration contemplated in the last term of the equation is *an integration extended over the surface of the volume.*

In order to give an interpretation to the equation let us write, as abbreviations,

$$U = \frac{\mu}{8\pi} H^2 + \frac{\epsilon}{8\pi} E^2 \tag{15}$$

and

$$\mathbf{s} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \tag{16}$$

Then equation (14) becomes

$$\partial P = -\dot{U}\partial\tau - \int \mathrm{s}_n dS \tag{17}$$

in which ∂P is the power, or energy per second, converted into heat or other form of energy within the element $\partial\tau$. This power is the sum of two terms, both with negative signs. We, therefore, naturally look to these terms as the source of supply of the power that is converted. One of the terms is a volume term and, taken with its negative sign, it may be regarded as the time rate of decrease of the magnetic and electrical energy in the element of volume, so that

$$U = \text{energy per unit volume.}$$

The other term is a surface term, and taken with its negative sign, it is the time rate at which energy flows into the element through its surface. Then $\mathrm{s}_n dS$ is the quantity of energy per second flowing through dS in the direction of the outward normal, that is,

$\mathbf{s}$ = energy per second flowing in the direction of $\mathbf{s}$ per unit area perpendicular to $\mathbf{s}$.

This vector s, defined in equation (16) is called *Poynting's Radiation Vector*, and was discovered by Professor J. H. Poynting.[1]

The equation (15) for the energy density in an electromagnetic field, and the equation (16) for the flow of electromagnetic energy per second per unit cross section of the energy beam are very important quantities in the theory of electric waves.

Although we have employed in the above derivation the general case in which there is an electric current of density $\mathbf{u}$ at the point x, y, z, it is seen that the whole demonstration holds when $\mathbf{u} = 0$.

[1] J. H. Poynting, *Phil. Trans.*, 2, p. 343, 1884.

We should then have $\partial P = 0$, and

$$\dot{U}\partial\tau = -\int \mathrm{s}_n dS \tag{18}$$

This means that in this special case that the rate of gain of electrical and magnetic energy within the region is equal to the rate at which electromagnetic energy flows in through the surface.

CHAPTER IV

WAVE EQUATIONS. PLANE WAVE SOLUTION

34. Digression to Find Curl Curl A.—In proper combinations of Maxwell's equations the work may be simplified by the use of a proposition in vector analysis concerning the curl of the curl of a vector. Let us designate the vector by **A**. Then let us perform elementary operations as follows:

$$\begin{aligned}\text{curl}_x \text{curl}\,\mathbf{A} &= \frac{\partial}{\partial y}\text{curl}_z\,\mathbf{A} - \frac{\partial}{\partial z}\text{curl}_y\,\mathbf{A}\\ &= \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\\ &= -\frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right).\end{aligned}$$

Subtracting and adding $\frac{\partial^2 A_x}{\partial x^2}$, we have

$$\text{curl}_x \text{curl}\,\mathbf{A} = -\nabla^2 A_x + \frac{\partial}{\partial x}(\text{div. }\mathbf{A}) \tag{1}$$

where as an abbreviation we have written the Laplacian operator

$$\nabla^2 A_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \tag{2}$$

In like manner,

$$\text{curl}_y \text{curl}\,\mathbf{A} = -\nabla^2 A_y + \frac{\partial}{\partial y}(\text{div. }\mathbf{A}) \tag{3}$$

$$\text{curl}_z \text{curl}\,\mathbf{A} = -\nabla^2 A_z + \frac{\partial}{\partial z}(\text{div. }\mathbf{A}) \tag{4}$$

These three curl curl components may be collected into a single vector equation by multiplying respectively by **i**, **j**, and **k** and adding, with the result

$$\text{curl curl}\,\mathbf{A} = -\nabla^2\mathbf{A} + \text{grad. div. }\mathbf{A} \tag{5}$$

where if ψ is any scalar quantity, and if **i**, **j**, **k** are unit vectors

along the three axes, then gradient ψ, which is abbreviated "grad ψ," is defined by the equation

$$\text{grad}\,\psi = \frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k} \tag{6}$$

and triangle square of a vector $\mathbf{A}$ is defined by

$$\begin{aligned} \nabla^2\mathbf{A} &= \nabla^2 A_x\mathbf{i} + \nabla^2 A_y\mathbf{j} + \nabla^2 A_z\mathbf{k} \\ &= \nabla^2\mathbf{A}_x + \nabla^2\mathbf{A}_y + \nabla^2\mathbf{A}_z \end{aligned} \tag{7}$$

35. Elimination Among the Electromagnetic Field Equations for a Homogeneous Isotropic Medium.—In a homogeneous medium ϵ, μ, and γ are constants. If the medium is isotropic, these quantities are also independent of direction. Under these conditions, Maxwell's Equations (A) and (B), Art. **30** may be written

$$\frac{4\pi\gamma\mathbf{E}}{c} + \frac{\epsilon\dot{\mathbf{E}}}{c} = \text{curl}\,\mathbf{H} \tag{8}$$

$$-\frac{\mu\dot{\mathbf{H}}}{c} = \text{curl}\,\mathbf{E} \tag{9}$$

If now we take the curl of both sides of (8), we have

$$\frac{4\pi\gamma}{c}\,\text{curl}\,\mathbf{E} + \frac{\epsilon}{c}\frac{\partial}{\partial t}(\text{curl}\,\mathbf{E}) = \text{curl curl}\,\mathbf{H}.$$

Replacing in this equation curl $\mathbf{E}$ by its value from (9), and replacing curl curl $\mathbf{H}$ by its value from (5), we have, since div. $\mathbf{H} = 0$,

$$\frac{4\pi\gamma\mu}{c^2}\frac{\partial\mathbf{H}}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla^2\mathbf{H} \tag{10}$$

Now starting with the other Maxwellian Equation (9) and taking the curl of both sides of it, we obtain

$$-\frac{\mu}{c}\frac{\partial}{\partial t}(\text{curl}\,\mathbf{H}) = \text{curl curl}\,\mathbf{E}$$

from which by (8) and (5) we obtain

$$\frac{4\pi\gamma\mu}{c^2}\frac{\partial\mathbf{E}}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla^2\mathbf{E} - \frac{4\pi}{\epsilon}\,\text{grad}\,\rho \tag{11}$$

The equations (10) and (11) are vector equations and are true for each of the components of $\mathbf{H}$ or $\mathbf{E}$, in a homogeneous

isotropic medium. For example, taking the x-components we have, after dividing by **i**, the scalar relations

$$\frac{4\pi\mu\gamma}{c^2}\frac{\partial H_x}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2 H_x}{\partial t^2} = \nabla^2 H_x \tag{12}$$

$$\frac{4\pi\mu\gamma}{c^2}\frac{\partial E_x}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \nabla^2 E_x - \frac{4\pi}{\epsilon}\frac{\partial \rho}{\partial x} \tag{13}$$

Similar expressions for the other components may be had by advancing the letters. Each component is thus obtainable in a differential equation not involving the other components, so that the problem may be completely solved in any cases in which the differential equation of the type of (12) or (13) can be solved. We shall not at present enter into the discussion of the general equations but shall consider certain important special cases.

36. Special Case in Which the Homogeneous Medium is an Insulating Medium Uncharged.—In this case the conductivity γ is zero and the intrinsic charge density ρ is also zero, so that each component of electric and magnetic intensity E_x, E_y, E_z, H_x, H_y, H_z satisfies an equation of the form

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 M}{\partial t^2} = \nabla^2 M \tag{14}$$

where M is a generic expression for either of the components of electric or magnetic intensity. This equation is of a type known in elasticity theory as the *wave equations.*

37. Special Case of a Plane Wave in an Insulating Homogeneous Uncharged Medium.—The equation (14) applies to this case, but this equation is to be still further specialized by making M a function of s and t alone,

$$M = f(s, t) \tag{15}$$

where

$$s = lx + my + nz \tag{16}$$

where l, m, n are the direction cosines of s, so that

$$\left.\begin{aligned} l &= \text{cos. of angle between } s \text{ and } x \\ m &= \text{cos. of angle between } s \text{ and } y \\ n &= \text{cos. of angle between } s \text{ and } z \\ 1 &= l^2 + m^2 + n^2 \end{aligned}\right\} \tag{17}$$

Equation (16) is the equation of all points x, y, z on a plane PQ (Fig. 1) perpendicular to s at its end; so that s is the perpendicular distance from the origin 0 to the plane.

For a fixed value of s, and at a fixed time, the value of M (15) is the same at all points of the plane. M is a generic symbol for each component of electric or magnetic intensity, so that each of these intensities is the same all over the plane s at a given time. As the time changes, these values of intensity in the plane change but remain of uniform value over the plane.

If on the other hand, the time is considered fixed, and different values are given to s, each of the different values of s will represent a different one of a series of parallel planes perpendicular to s, and over each of these different planes the intensity will be uniform but different from plane to plane.

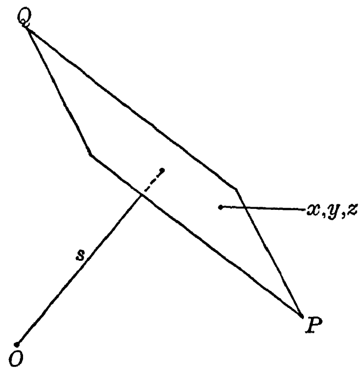

FIG. 1.—Every point x, y, z, of the plane $Q\,P$ satisfies equation (16), in which s is the length of the perpendicular from O to the plane

The field of electric and magnetic intensity may thus be called a *plane field.*

In the case of the plane field the wave equation (14) reduces to a simpler form if we express $\nabla^2 M$ in terms of s thus:

$$\frac{\partial M}{\partial x} = \frac{\partial M}{\partial s}\frac{\partial s}{\partial x} = l\,\frac{\partial M}{\partial s},$$

Likewise,

$$\left.\begin{aligned} \frac{\partial^2 M}{\partial x^2} &= l^2\,\frac{\partial^2 M}{\partial s^2}. \\ \frac{\partial^2 M}{\partial y^2} &= m^2\,\frac{\partial^2 M}{\partial s^2} \\ \frac{\partial^2 M}{\partial z^2} &= n^2\,\frac{\partial^2 M}{\partial s^2} \end{aligned}\right\} \qquad (18)$$

Whence (14) becomes

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2 M}{\partial t^2} = \frac{\partial^2 M}{\partial s^2} \tag{19}$$

Each component of electric and magnetic intensity in the plane field satisfies an equation of the form of (19). This equation, for reasons that we shall soon see, is called *the Plane-wave Equation.*

38. Classification of Solutions of the Plane-wave Equation. Let us now undertake a solution of the plane-wave equation (19), in which M is a generic symbol for any of the electric or magnetic intensities.

Two classes of solutions will appear. These we shall call Class I and Class II, described as follows:

Class I. All solutions that reduce both sides of (19) to zero;

Class II. All solutions that reduce the two sides of (19) to equality but not to zero.

We shall find that only solutions of Class II are important for electric wave theory, but both classes will now be considered.

39. Solutions of Class I.—Let P be any solution of (19) of Class I. Then by definition of this class

$$\frac{\partial^2 P}{\partial t^2} = 0 \tag{20}$$

$$\frac{\partial^2 P}{\partial s^2} = 0 \tag{21}$$

An integration of these equations as simultaneous gives

$$P = A + Bt + Cs + Dst \tag{22}$$

in which A, B, C, D are constants of integration. In all cases in which the intensities are restricted to finite values $B = C = D = 0$, and

$$P = A \tag{23}$$

This constant A is also zero in all cases in which only fluctuating quantities enter into consideration.

40. Solutions of Class II.—Returning now to the plane-wave equation (19), let us seek for solutions of Class II; that is, for solutions that do not reduce the two sides of the equation to zero.

Any function of $s + at$ (if a has the proper value) is a solution of (19), as may be seen by direct substitution as follows:

Let

$$M = G_{(s+at)} \tag{24}$$

where G is a symbol for "function," and let us take the second derivatives of G with respect to s and t. For this purpose, let us designate the first and second derivatives of G with respect to its argument $(s + at)$ by G' and G'' respectively. Then

$$\frac{\partial M}{\partial t} = aG', \qquad \frac{\partial M}{\partial s} = G',$$

$$\frac{\partial^2 M}{\partial t^2} = a^2G'', \qquad \frac{\partial^2 M}{\partial s^2} = G'';$$

whence, equation (19) becomes

$$\frac{\mu\epsilon}{c^2} a^2G'' = G'' \tag{25}$$

This equation is satisfied by $G'' = 0$, which has already been treated in Class I. It is also satisfied by any function G whatever, provided

$$\frac{\mu\epsilon}{c^2} a^2 = 1.$$

That is,

$$a = +\frac{c}{\sqrt{\mu\epsilon}} \tag{26}$$

or

$$a = -\frac{c}{\sqrt{\mu\epsilon}}.$$

It thus appears that in our attempt to find one functional solution of (19) we have found two; namely,

$$M = F_{\left(s - \frac{c}{\sqrt{\mu\epsilon}} t\right)}$$

and

$$M = G_{\left(s + \frac{c}{\sqrt{\mu\epsilon}} t\right)}$$

where F and G are *any functions* of their respective arguments.

Now since equation (19) is linear and homogeneous, the sum of the several solutions is a solution; that is

$$M = F_{\left(s - \frac{c}{\sqrt{\mu\epsilon}} t\right)} + G_{\left(s + \frac{c}{\sqrt{\mu\epsilon}} t\right)} + P \tag{27}$$

This solution is the complete integral of equation (19); for the term P includes all solutions that satisfy (19) by reducing

both sides to zero, and the terms F and G being two arbitrary functions include all other solutions of the second-order partial differential equation with two independent variables.

If we omit the P solution, which as we have shown in Art. **39**, is of no importance where only fluctuating intensities enter into consideration, we shall have only the F and G solutions of (27).

41. Examination of the Plane-wave Solution. Velocity.—In equation (27) is given the complete solution of the plane-wave equation (19). In this solution M is any one of the components of electric or magnetic intensity. The functions F and G may be different for the different components, but the arguments of these two functions remain always the same two arguments.

The several functions are interrelated by Maxwell's equations and are further delimited by the boundary conditions at the source of the disturbance and at any surfaces of discontinuity that exist between different media.

Without at present entering into these interrelations and limitations, we can discover certain interesting properties of the field by examining the general solution (27). We can, for example, obtain the result that the F and G parts of the field are disturbances that move with a finite velocity, and we can determine the velocity as follows:

Let us confine our attention at first to the function F, and write

$$M = F\left(s - \frac{c}{\sqrt{\mu\epsilon}}t\right)$$

We see that, whatever value M may happen to have all over the plane at the distance s_1 from the origin at the time t_1, it will have the same value all over the plane s_2 at the time t_2, provided

$$s_1 - \frac{c}{\sqrt{\mu\epsilon}}t_1 = s_2 - \frac{c}{\sqrt{\mu\epsilon}}t_2 \tag{28}$$

for then

$$F\left(s_1 - \frac{c}{\sqrt{\mu\epsilon}}t_1\right) = F\left(s_2 - \frac{c}{\sqrt{\mu\epsilon}}t_2\right)$$

That is, the time at which a given condition exists at two different distances are related to the distances by the equation (28) or

$$\frac{s_2 - s_1}{t_2 - t_1} = \frac{c}{\sqrt{\mu\epsilon}}.$$

The distance traveled $s_2 - s_1$ divided by the time to travel it $(t_2 - t_1)$ gives the velocity; whence

$$v = \frac{c}{\sqrt{\mu\epsilon}} \tag{29}$$

is the velocity with which the condition at s_1 moves in the direction of increasing s.

In a similar way it may be shown that the equation

$$M = G\left(s + \frac{c}{\sqrt{\mu\epsilon}}\,t\right)$$

represents a condition moving in the opposite direction (the direction of decreasing s) with the same velocity.

From the above discussion it appears that if we have an electromagnetic field in which all of the components of electric and magnetic intensity are functions of s and t alone, where s is the perpendicular distance from an arbitrary origin, and if the intensities are supposed to remain everywhere and at all times finite, and if there are no constant components of intensity, the quantity P becomes 0, and each component of intensity consists in general of two superposed disturbances, or waves, moving in opposite directions along the axis of s with the velocity given in (29).

The form of the functions F and G will depend upon the manner of the origination of the disturbance and upon the conditions at certain surfaces of discontinuity bounding the homogeneous region under consideration. In particular cases one of the functions G, say, may be everywhere zero, and the whole field will move forward in one direction with the velocity v. In other particular cases, as when we have a reflection of waves, both the forward-moving wave and the backward-moving wave will coexist and give an interference system. The importance of having the two functions in the solution is precisely this—that it enables us to give a description of the phenomena of reflection when they occur.

42. Velocity in Free Space Equals the Ratio of Units, Equals the Velocity of Light.—In space devoid of all matter, $\epsilon = \mu = 1$; therefore, the velocity (29) becomes in empty space

$$v_0 = c,$$

where c is the number of absolute c.g.s. electrostatic units of quantity of electricity in one electromagnetic unit of quantity.

The prediction that electric waves in free space should have the value here given was made by Maxwell in his original writings on the electromagnetic theory of light. Before that time it was known from experiments that c, the ratio of the units, was approximately the velocity of light. Maxwell himself made some of the measurements of the ratio of the units. Later experimental determinations of these quantities are given in the following table.

Table I.—Comparison of Velocity of Light with Ratio of Units[1]

	Velocity of light	Observer
	2.99853×10^{10} cm./sec..	Michelson
	2 99860	Newcomb
	2 99860	Perrotin
	2 99852	Weinberg
Average	2 99856	

	Ratio of units	Observer
	3.0057×10^{10}. . . .	Himstedt
	3 0000 . . .	Rosa
	2 9960 .. .	Thomson and Searle
	2.9913 . .	H. Abraham
	3 0010 . .	Hurmuzescu
	2 9978	Perot and Fabry
	2 9971	Rosa and Dorsey
Average	2 9984	

44. Refractive Index for Electric Waves.—To get the index of refraction for electric waves of any insulating medium of dielectric constant ϵ and permeability μ, it is only necessary to note that the velocity in this medium is

$$v = \frac{c}{\sqrt{\epsilon\mu}}$$

while the velocity in vacuo is

$$v_0 = c.$$

The ratio of these two velocities is the index of refraction n of the medium for the particular frequency at which ϵ and μ are measured; that is,

$$n = v_0/v = \sqrt{\epsilon\mu} \tag{30}$$

[1] For references to literature see Rosa and Dorsey: *Bulletin Bureau of Standards*, Vol. 3, Nos. 3 and 4, 1907.

It is to be noted that the derivation of this equation assumes that the medium is non-conductive and that there are no motions of charged particles within the medium; for such a motion constitutes a current, and all such currents have been excluded from the special problem of the insulating medium.

45. The Plane Electric Wave in a Non-crystalline Homogeneous Dielectric is a Transverse Wave, with its Electric and Magnetic Intensities Perpendicular to the Direction of Propagation and Perpendicular to Each Other.—*Proof:* Each component of electric intensity of the wave moving in direction of positive s is a function of $s - vt$, and therefore of $t - s/v$, where

Let

$$\left.\begin{aligned} v &= \frac{c}{\sqrt{\mu\epsilon}} \\ s &= lx + my + nz. \\ \\ E_x &= f(t - s/v). \\ E_y &= g(t - s/v) \\ E_z &= h(t - s/v) \end{aligned}\right\} \qquad (31)$$

where f, g, and h are any functions of their argument $t - s/v$.

Let the derivatives of f, g, h with respect to $t - s/v$ be indicated by f', g', h', and let us now determine the values of the components of **H** by Art. **30** Equations (B), the x-component of which gives

$$-\frac{\mu}{c}\frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$= -\frac{m}{v}h' + \frac{n}{v}g'.$$

Integrating and multiplying by $-\frac{c}{\mu}$, we have (omitting the constant of integration as of no significance for the wave-field)

Likewise

$$\left.\begin{aligned} H_x &= \sqrt{\frac{\epsilon}{\mu}}(mh - ng) \\ &= \sqrt{\frac{\epsilon}{\mu}}(mE_z - nE_y) \\ \\ H_y &= \sqrt{\frac{\epsilon}{\mu}}(nE_x - lE_z) \\ H_z &= \sqrt{\frac{\epsilon}{\mu}}(lE_y - mE_x) \end{aligned}\right\} \qquad (32)$$

Let us now recall that l, m, and n are the direction-cosines of s; that is, l, m, and n are the components along the axes of x, y, and z of a unit vector $\mathbf{U}_s$ along s; whence by (5), Art. **31,** equations (32) may be combined into the vector equation

$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}}\,\mathbf{U}_s \times \mathbf{E} \tag{33}$$

where

$\mathbf{U}_s$ = a unit vector in direction of propagation s.

This equation (33) *gives the magnetic intensity* $\mathbf{H}$ *in magnitude and direction in terms of the electric intensity* $\mathbf{E}$ *for the case of a plane wave traveling in the direction* s (*or* $\mathbf{U}_s$) *in a homogeneous insulating medium*

In *magnitude*, it is seen by (33)

$$H = \sqrt{\frac{\epsilon}{\mu}}E \tag{34}$$

In direction $\mathbf{H}$ is $\perp$ to $\mathbf{E}$ and $\perp$ to s.

To prove completely the proposition enunciated in the heading of this section, it remains to prove $\mathbf{E}$ also perpendicular to s. This can be done by starting with H_x, H_y, and H_z as functions of $(t - s/v)$. The equations will be similar to (31) but with different functions. Then applying Maxwell's equations (A), Art. **30,** of the type

$$\frac{\epsilon}{c}\frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

we obtain

$$E_x = -\sqrt{\frac{\mu}{\epsilon}}\left(mH_z - nH_y\right)$$

and similar equations for E_y and E_z; whence vectorially

$$\mathbf{E} = -\sqrt{\frac{\mu}{\epsilon}}\mathbf{U}_s \times \mathbf{H} = \sqrt{\frac{\mu}{\epsilon}}\mathbf{H} \times \mathbf{U}_s \tag{35}$$

This equation agrees with (33) and shows in addition that $\mathbf{E}$ is $\perp$ to $\mathbf{U}_s$. *The conclusion from* (33) *and* (35) *is then that* $\mathbf{E}$, $\mathbf{H}$, *and* $\mathbf{U}_s$ *are mutually perpendicular and are oriented with respect to one another in the same way as the axes* x, y, z, *in Fig.* 3, *Art.* **19.**

The direction of propagation, which is the direction of $\mathbf{U}_s$, is also the direction of Poynting's vector $\mathbf{s}$.

Various nemonic rules have been suggested for remembering the orientation of **E**, **H**, and s. A simple one is as follows:

$$\mathbf{E} = \text{east}, \quad \text{s} = \text{south}, \quad \mathbf{H} = \text{high (upward)}.$$

For the backward moving wave the rule for the orientation of the intensities with respect to the direction of propagation is the same; namely, equation (35).

It is seen, however, that if we reverse the direction of one of the quantities **E, H, s,** *we must reverse one other of them but not both,* since any one of the vector quantities has as a factor the vector product of the other two.

46. The Instantaneous Electric Energy per Unit Volume is Equal to the Instantaneous Magnetic Energy per Unit Volume of a Single Plane Wave.—This proposition follows at once, by squaring (34) and dividing by $\frac{8\pi}{\mu}$, which gives

$$\frac{\epsilon}{8\pi} E^2 = \frac{\mu}{8\pi} H^2 \tag{36}$$

This equation, as well as (34) from which it is derived, holds true when there is a single plane wave moving in one direction. It does not hold when there exists an interference system, as will be shown below.

47. Harmonic Solution for a Plane Wave, Plane Polarized, in a Homogeneous Insulator.—Up to the present we have treated the problem of the plane wave by means of general functions, and we have shown that the electric and magnetic intensities and the direction of propagation are mutually perpendicular.

Let us assume that the wave is *plane polarized.* This means that the direction of the electric and magnetic intensities do not change. We may choose the axes so that **E** is along the x-axis, and **H** is along the y-axis, then the direction of propagation will be the direction of the z-axis; and we may write

$$E_x = f(t - z/v) \tag{37}$$

and by (34)

$$H_y = \sqrt{\frac{\epsilon}{\mu}} f(t - z/v) \tag{38}$$

where

$$v = \frac{c}{\sqrt{\mu\epsilon}} \tag{39}$$

It is now proposed to limit the problem by assuming that the electric intensity E_x is a harmonic function of the time. By (37) it will then be a harmonic function of $(t - z/v)$, and may be writtten

$$E_x = E \sin\{\omega(t - z/v) + \phi\} \tag{40}$$

and by (38),

$$H_y = \sqrt{\frac{\epsilon}{\mu}} E \sin\{\omega(t - z/v) + \phi\} \tag{41}$$

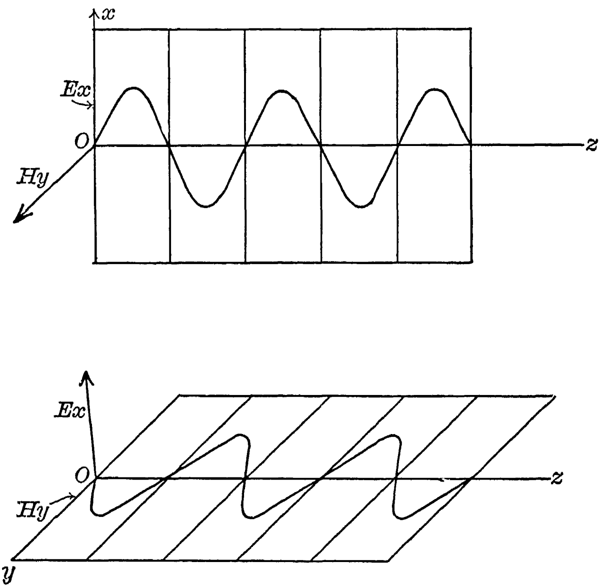

FIG 2.—Orientation of electric and magnetic intensities in a plane wave traveling in the z-direction in a homogeneous isotropic medium.

where

E = amplitude of E_x;
ω = angular velocity in radians per second of the harmonic oscillation $= 2\pi/T$, where
T = period;
ϕ = phase angle depending on choice of origin of time.

Equations (40) and (41) give the electric and magnetic intensities of a harmonic wave moving in the z-direction. It is seen that *in such a wave the electric and magnetic intensities are in phase in time and space.* At a given time the distribution of intensities for different values of z are given in Fig. 2; where,

for simplicity of drawing, separate diagrams are made for the two intensities.

To obtain the wave length λ, we have only to note that the addition of λ to z does not change E_x or H_y in (40) and (41). This means that

$$\frac{\omega\lambda}{v} = 2\pi$$

or

$$\lambda = \frac{2\pi v}{\omega} = vT \tag{42}$$

As we have shown in the examination of the general functions of $(t - s/v)$, the whole diagrams of Fig. 2, except the axial line oz, are supposed to move forward in the z-direction with the velocity v.

If the observation is made at a fixed point on the axis, $z =$ constant, the vectors of electric and magnetic intensity will fluctuate sinusoidally with the time.

The plane of the wave is a plane perpendicular to oz and any such plane has all over it a uniform value of electric intensity, and of magnetic intensity, at a given time.

CHAPTER V

REFLECTION OF A PLANE WAVE FROM A PERFECT CONDUCTOR

In the present chapter we shall treat the reflection of a plane electric wave from the surface of a perfect conductor. In Arts. **48** and **49** the wave will be considered to be harmonic and to be incident normally. In Arts. **50** and **51** the more general case will be considered, in which the incidence is oblique and the wave not limited to the harmonic form.

In a later chapter cases in which the conductor is not a perfect conductor will be considered.

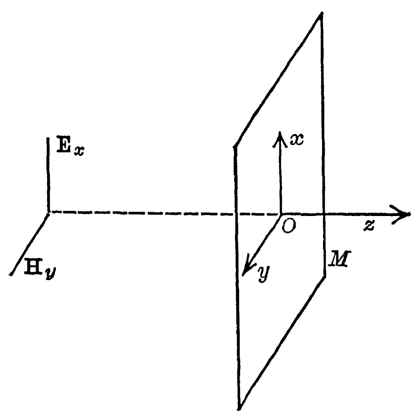

FIG. 1.—Electric wave E_x, H_y, traveling in the z-direction, incident normally on a perfectly conductive surface M.

48. Reflection of a Harmonic Plane-polarized Plane Wave from a Perfectly Conductive Plane at Normal Incidence.—Let M, Fig. 1, be a perfect conductor with a plane surface in the xy-plane through the origin of coördinates. Let a plane-polarized wave coming from the left of the surface in a dielectric medium of dielectric constant ϵ and permeability μ be incident normally upon the surface, and let us choose the axes so that the x-axis is in the direction of the electric intensity, and the y-axis in the direction of the magnetic intensity.

The characteristic of a perfect conductor is that the electric intensity within the conductor is zero. In the medium in contact with the conductor the *tangential component of electric* intensity is continuous with its value within the conductor, and therefore zero at all times.

We have assumed the incident wave harmonic, but a single harmonic value for E_x, such as is given in (40), Art. **47**, does not possess the property of being zero at $z = 0$, and is therefore insufficient to represent the system of waves in the present problem. By our general solution (27), Art. **40**, we may add to the wave traveling in the z-direction another wave traveling in the opposite direction, and with proper choice of intensities, phases, etc., it is possible to make the direct and the reflected waves annul each other as to electric intensity at the surface of the conductor. Since the incident wave is harmonic, the reflected wave to annul it must be also harmonic and of the same frequency and same phase angle. By proper choice of the origin of time we may make this phase angle $\phi = 0$, and write the solution

$$E_x = E_1 \sin \omega(t - z/v) + E_2 \sin \omega(t + z/v) \tag{1}$$

Now by the condition at the surface, we have, when $z = 0$, $E_1 = -E_2 = E$ (say). Therefore,

$$E_x = E \sin \omega(t - z/v) - E \sin \omega(t + z/v) \tag{2}$$

The second term has its direction of propagation and also its intensity reversed with respect to the first term; whence, by Art. **45**, Chapter IV, it is seen that the corresponding amplitude of H for the second term will have the same direction as the amplitude of H for the first term, and by (32), Art. **45**, we shall have

$$H_y = \sqrt{\frac{\epsilon}{\mu}}\, E \sin \omega(t - z/v) + \sqrt{\frac{\epsilon}{\mu}}\, E \sin \omega(t + z/v) \tag{3}$$

This equation for H_y may, if desired, be independently derived by substituting the value (2) for E_x, with E_y and E_z equal zero, into Maxwell's Equation (B), Art. **30**.

Equations (2) and (3) show that the magnetic intensity H_y is made up of two harmonic wave-trains traveling in opposite directions, having equal amplitudes, and having the reflected magnetic intensity in phase with the incident magnetic intensity; while the electric intensity E_x consists also of a direct and a

reflected wave of equal amplitude, but the reflected electric intensity is opposite in phase to the incident electric intensity.

Let us now put equations (2) and (3) into a better form for their interpretation. Expanding the sine terms by the trigonometric formulas for the sine of a sum or a difference, we obtain

$$E_x = -2E \cos \omega t \sin (\omega z/v) \tag{4}$$

$$H_y = 2\sqrt{\frac{\epsilon}{\mu}}\, E \sin \omega t \cos (\omega z/v) \tag{5}$$

49. Plot of Stationary Wave System.—A plot of the two intensities is given in Fig. 2, where, to obviate difficulty in plotting,

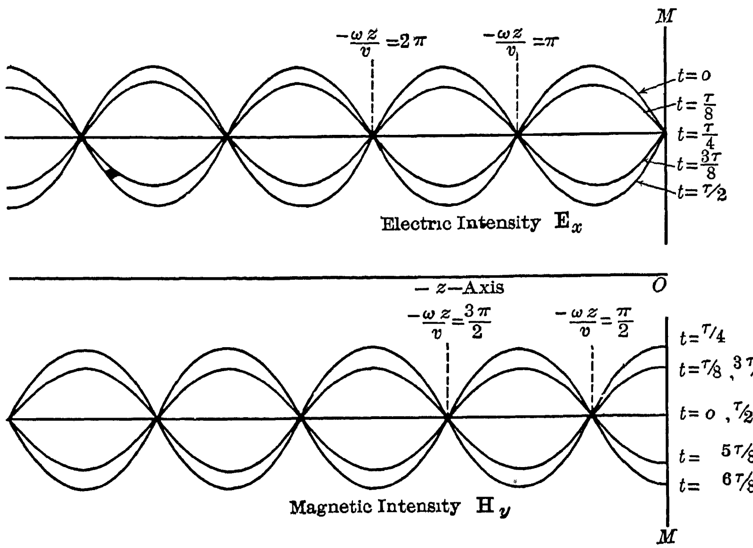

FIG. 2—Stationary waves of electric and magnetic intensity at normal incidence on a perfectly conductive plane surface M. In the figure the period is represented as τ.

no effort is made to show that the magnetic and electric intensities are at right angles to each other.

The equations (4) and (5) are thus seen to be the equations to two *stationary wave systems*. There are certain points in space where the electric intensity is always zero and certain other points where the magnetic intensity is always zero. These positions of constant zero-intensity are called *nodes*. Between the electric nodes and between the magnetic nodes there are points of maximum fluctuation of intensity, which are called *loops*.

Whereas, in the single free train of waves the electric and magnetic intensities are exactly in phase in time and space; in the interference system, or stationary system, the electric and magnetic intensities are 90° out of phase in time and space.

The wavelength in the incident wave by (42), Art. **47,** is $\lambda = 2\pi v/\omega$. The positions of the nodes in the stationary system are seen to be at the following values of z:

The nodes of E_x are at

$$-z = 0, \quad \pi v/\omega, \quad 2\pi v/\omega, \quad 3\pi v/\omega \quad \text{etc.};$$

that is

$$-z = 0, \quad \lambda/2, \quad \lambda, \quad 3\lambda/2, \quad \text{etc.} \tag{6}$$

The nodes of H_y are at

$$-z = \lambda/4, \quad 3\lambda/4, \quad 5\lambda/4, \quad 7\lambda/4, \quad \text{etc.} \tag{7}$$

Loops exist halfway between these respective nodes.

It is seen that the distance between consecutive electric nodes or consecutive magnetic nodes is half the wavelength of the incident wave. The distance between consecutive electric loops or consecutive magnetic loops is the same distance.

Since the reflected intensities are equal to the incident intensities in amplitude, the perfectly conductive surface is a perfect reflector for electromagnetic waves.

50. Reflection of a Plane Wave from a Perfectly Conductive Plane at Arbitrary Incidence.—Let the conductive plane, which we shall call the *mirror*, pass through the origin of coördinates and be perpendicular to the x-axis. Suppose a plane direct wave to be traveling in a medium of dielectric constant ϵ and permeability μ, and in the direction of a line s with direction cosines l, m, n. Then any point x, y, z on the incident wave front W, Fig. 2, will satisfy the equation

$$lx + my + nz = s \tag{8}$$

where s is the distance from 0 to W,

In the *Direct Wave*, let the components of electric intensity by any functions f, g, h of $(t - s/v)$; that is

$$\left.\begin{aligned} E_x &= f(t - s/v) \\ E_y &= g(t - s/v) \\ E_z &= h(t - s/v) \end{aligned}\right\} \tag{9}$$

where

$$v = \frac{c}{\sqrt{\mu\epsilon}} \tag{10}$$

Then as in equation (32), Art. **45**,

$$\left.\begin{aligned} H_x &= \sqrt{\frac{\epsilon}{\mu}}(mE_z - nE_y) \\ H_y &= \sqrt{\frac{\epsilon}{\mu}}(nE_x - lE_z) \\ H_z &= \sqrt{\frac{\epsilon}{\mu}}(lE_y - mE_x) \end{aligned}\right\} \tag{11}$$

It is apparent that this direct wave alone is not sufficient, for the reason that the tangential components of electric force must be at all times zero at the mirror, and the values of (9) do not satisfy this condition. It is, therefore, necessary to suppose a reflected wave also to exist and to be superposed upon the direct wave.

We shall assume the reflected wave to be also a *plane* wave and to be traveling in some unknown direction along a line s_1, with direction cosines l_1, m_1, n_1, and shall show that with proper choice of s_1 and with proper intensities in the reflected wave, the proper boundary conditions are satisfied.

The reflected wave may be expressed in terms of arbitrary functions f_1, g_1 and h_1 as follows:

$$\left.\begin{aligned} E_{1x} &= f_1(t - s_1/v) \\ E_{1y} &= g_1(t - s_1/v) \\ E_{1z} &= h_1(t - s_1/v) \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} H_{1x} &= \sqrt{\frac{\epsilon}{\mu}}(m_1E_{1z} - n_1E_{1y}) \\ H_{1y} &= \sqrt{\frac{\epsilon}{\mu}}(n_1E_{1x} - l_1E_{1z}) \\ H_{1z} &= \sqrt{\frac{\epsilon}{\mu}}(l_1E_{1y} - m_1E_{1x}) \end{aligned}\right\} \tag{13}$$

with

$$l_1x + m_1y + n_1z = s_1 \tag{14}$$

where s_1 is the distance OW_1.

Now by the conditions at the mirror, when we put $x = 0$, the total tangential electric force must be zero; that is, the sum of the direct and the reflected E_y and E_z values must be zero; hence

$$\left.\begin{aligned} 0 &= g\left(t - \frac{my_0 + nz_0}{v}\right) + g_1\left(t - \frac{m_1y_0 + n_1z_0}{v}\right) \\ 0 &= h\left(t - \frac{my_0 + nz_0}{v}\right) + h_1\left(t - \frac{m_1y_0 + n_1z_0}{v}\right) \end{aligned}\right\} \tag{15}$$

where in these equations y_0 and z_0 are coördinates of any point in the surface of the mirror. To make (15) true for all such points and for all values of t, we must have for the operators g and g_1, h and h_1, the relations

$$\left.\begin{aligned} g_1 &= -g \\ h_1 &= -h \end{aligned}\right\} \tag{16}$$

and for the direction cosines,

$$\left.\begin{aligned} m_1 &= +m \\ n_1 &= +n \end{aligned}\right\} \tag{17}$$

Let us determine the other direction cosine l_1. By the fact that the sum of the squares of the direction cosines of a given line is unity, l_1 is equal to plus or minus l; but if it were plus l, then s_1 would be identical with s for any given point x, y, z and the total y and z-components of E would by (16) be zero everywhere at all times, and our incident wave would have only an

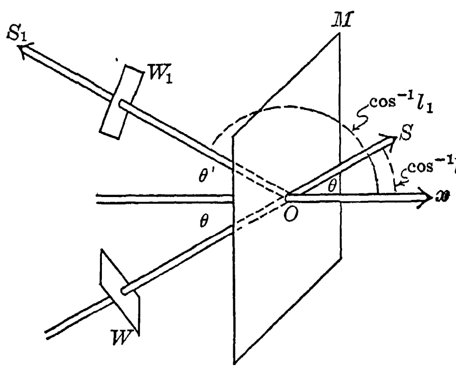

FIG. 3 —Illustrating a plane wave W traveling in direction S incident at angle of incidence θ upon a plane perfectly conductive surface M.

x-component and would be traveling parallel to the mirror. This case is of no interest, as the problem is then, so far as concerns the dielectric medium, the same as that with the mirror absent. Excluding this case, equivalent to no mirror present, we have in all other cases

$$l_1 = -l \tag{18}$$

The equations (17) and (18) show that the electric radiation obeys the ordinary law of reflection of light; namely, *the reflected ray is in the same plane with the incident ray and the normal to the mirror at the point of incidence, and the angle of reflection is equal to the angle of incidence.* (Proof follows.)

This is seen by reference to Fig. 3. The angle of incidence $\Theta = \cos^{-1} l$. The angle of reflection $\Theta' =$ the supplement of $\cos^{-1} l_1 = \Theta$. The equality of m_1 to m and of n_1 to n, makes the incident and reflected beam in the same plane perpendicular to the mirror.

Returning now to the question of electric and magnetic intensities, we have found the form of g_1 and h_1 in terms of g and h. It remains to find the form of f_1. This can be done by employing the fact that the electric intensity is in the wave front in both the direct and reflected waves; that is, the components in the directions s and s_1 are respectively zero. This means that

$$lf + mg + nh = 0 \tag{19}$$

$$l_1 f_1 + m_1 g_1 + n_1 h_1 = 0 \tag{20}$$

In view of (16), (17) and (18) the equation (20) becomes

$$-lf_1 - mg - nh = 0,$$

which added to (19) gives

$$f_1 = f \tag{21}$$

51. Intensities in Direct Wave and Reflected Wave, and Total Intensities at the Mirror.—Summarizing the results, we have for the intensities of the direct and reflected waves and for the total intensities at the mirror the following equations:

Direct Wave

$$\left.\begin{aligned} E_x &= f(t - s/v) & H_x &= \sqrt{\frac{\epsilon}{\mu}}\,(mE_z - nE_y) \\ E_y &= g(t - s/v) & H_y &= \sqrt{\frac{\epsilon}{\mu}}\,(nE_x - lE_z) \\ E_z &= h(t - s/v) & H_z &= \sqrt{\frac{\epsilon}{\mu}}\,(lE_y - mE_x) \end{aligned}\right\} \tag{22}$$

Reflected Wave

$$\left.\begin{aligned} E_{1x} &= f(t - s_1/v) & H_{1x} &= \sqrt{\frac{\epsilon}{\mu}}\,(mE_{1z} - nE_{1y}) \\ E_{1y} &= -\,g(t - s_1/v) & H_{1y} &= \sqrt{\frac{\epsilon}{\mu}}\,(nE_{1x} + lE_{1z}) \\ E_{1z} &= -\,h(t - s_1/v) & H_{1z} &= \sqrt{\frac{\epsilon}{\mu}}(-lE_{1y} - mE_{1x}) \end{aligned}\right\} \tag{23}$$

Total Field at the Mirror by (9), (12), (16), (22) and (23)

$$\left.\begin{array}{ll} E_x + E_{1x} = 2E_x & H_x + H_{1x} = 0 \\ E_y + E_{1y} = 0 & H_y + H_{1y} = 2H_y \\ E_z + E_{1z} = 0 & H_z + H_{1z} = 2H_z \end{array}\right\} \text{at } x = 0 \qquad (24)$$

It is seen that the effect on the plane wave of the plane perfectly conductive mirror is to double the normal electric intensity at the mirror and annihilate the tangential electric intensities; also to annihilate the normal magnetic intensities and double the tangential magnetic intensities at the mirror.

In the space at any distance from the surface of the mirror the equations (22) and (23) permit the complete computation of the reflected wave in terms of the direct electric intensities where these are known.

CHAPTER VI

VITREOUS REFLECTION AND REFRACTION

52. Reflection and Refraction of a Plane Electric Wave by a Homogeneous Insulator.—Suppose a plane electric wave in an insulating medium of inductivity ϵ_1, and permeability μ_1 to be incident upon the plane surface of a second insulating medium of inductivity ϵ_2 and permeability μ_2.

Let the surface between the two media be through the origin of coördinates and perpendicular to the x-axis, as in Fig. 1. Let us assume that the direct wave is traveling in the direction of s_1 with direction cosines l_1, m_1, and n_1; and that there is a refracted wave traveling in the second medium in some direction s_2 (direction cosines l_2, m_2, n_2), and also a reflected wave in the first medium traveling in some direction s_3 (cosines l_3, m_3, n_3). The velocity of the waves in the first medium is

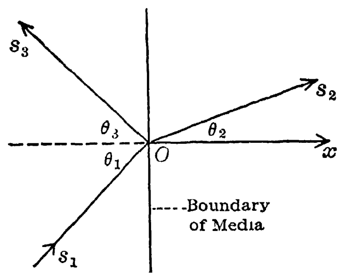

FIG. 1.—Concerning reflection and refraction at a boundary.

$$v_1 = \frac{c}{\sqrt{\mu_1\epsilon_1}} \tag{1}$$

the velocity in the second medium is

$$v_2 = \frac{c}{\sqrt{\mu_2\epsilon_2}} \tag{2}$$

It is required to find the directions of propagation of the reflected and refracted waves, and their intensities relative to the incident intensities.

The geometrical equations of the three wave-fronts are respectively

$$\left.\begin{aligned} l_1x + m_1y + n_1z &= s_1 \\ l_2x + m_2y + n_2z &= s_2 \\ l_3x + m_3y + n_3z &= s_3 \end{aligned}\right\} \tag{3}$$

We shall first write down the values of the electric intensities in the three waves respectively:

In the Direct Wave

$$\left.\begin{aligned} E_{1x} &= f_1(t - s_1/v_1) \\ E_{1y} &= g_1(t - s_1/v_1) \\ E_{1z} &= h_1(t - s_1/v_1) \end{aligned}\right\} \tag{4}$$

In the Refracted Wave

$$\left.\begin{aligned} E_{2x} &= f_2(t - s_2/v_2) \\ E_{2y} &= g_2(t - s_2/v_2) \\ E_{2z} &= h_2(t - s_2/v_2) \end{aligned}\right\} \tag{5}$$

In the Reflected Wave

$$\left.\begin{aligned} E_{3x} &= f_3(t - s_3/v_1) \\ E_{3y} &= g_3(t - s_3/v_1) \\ E_{3z} &= h_3(t - s_3/v_1) \end{aligned}\right\} \tag{6}$$

The magnetic intensities in these three waves are given respectively by the vector equations (cf. (33), Chapter IV):

$$\left.\begin{aligned} \mathbf{H}_1 &= \sqrt{\frac{\epsilon_1}{\mu_1}}\,\mathbf{U}_1 \times \mathbf{E}_1 \\ \mathbf{H}_2 &= \sqrt{\frac{\epsilon_2}{\mu_2}}\,\mathbf{U}_2 \times \mathbf{E}_2 \\ \mathbf{H}_3 &= \sqrt{\frac{\epsilon_1}{\mu_1}}\,\mathbf{U}_3 \times \mathbf{E}_3 \end{aligned}\right\} \tag{7}$$

where $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{U}_3$ are unit vectors in the directions of s_1, s_2, and s_3 respectively.

In addition to the above equations we have by equation (26), Chapter I, the condition that at the boundary between the two media the normal component of electric induction is continuous, since there is no intrinsic surface charge, and this gives

$$\epsilon_1 f_1\left(t - \frac{m_1 y + n_1 z}{v_1}\right) + \epsilon_1 f_3\left(t - \frac{m_3 y + n_3 z}{v_1}\right) = \epsilon_2 f_2\left(t - \frac{m_2 y + n_2 z}{v_2}\right) \tag{8}$$

This equation is true for all values of y and z in the surface between the two media; whence it follows that

$$\frac{m_1}{v_1} = \frac{m_3}{v_1} = \frac{m_2}{v_2} \tag{9}$$

$$\frac{n_1}{v_1} = \frac{n_3}{v_1} = \frac{n_2}{v_2} \tag{10}$$

Now it is to be noted that l_1 is the cosine of the angle of incidence of the ray $= \cos \Theta_1$; l_2 is the cosine of the angle of refraction $= \cos \Theta_2$; and l_3 is the cosine of the supplement of the angle of reflection $= -\cos \Theta_3$; whence

$$\sqrt{m_1^2 + n_1^2} = \sin \Theta_1 \tag{11}$$

$$\sqrt{m_2^2 + n_2^2} = \sin \Theta_2 \tag{12}$$

$$\sqrt{m_3^2 + n_3^2} = \sin \Theta_3 \tag{13}$$

And by taking the square root of the sum of the squares of (9) and (10) we obtain

$$\sin \Theta_1 = \sin \Theta_3 \tag{14}$$

$$\frac{\sin \Theta_1}{\sin \Theta_2} = \frac{v_1}{v_2} \tag{15}$$

Equation (14) shows that the angle of reflection is equal to the angle of incidence. Equation (15) shows that the ratio of the sine of the angle of incidence to the sine of the angle of refraction is the ratio of the velocity in the incident medium to the velocity in the refracting medium.

These are the ordinary laws of reflection and refraction. To make these laws complete we need also to show that the incident ray, the refracted ray, the reflected ray and the normal to the surface are in the same plane. This can be seen to be true by noticing that the y and z axes have not yet been chosen. If we make the z-axis perpendicular to the incident ray, n_1 will be zero; and by (10) n_2 and n_3 are also zero, so that all three of the rays are perpendicular to the z-axis, and are, therefore, in the same plane, which plane also contains x, since it is a concurrent perpendicular to z.

In order next to determine the coefficient of reflection of the surface between the media, let us keep the orientation of axis above suggested. Then the three rays are in the xy-plane, as shown in Fig. 2. Let us compare the energy incident per second upon any area dS with the energy transmitted through dS per

second. The cross sections of the three beams with their bases on dS respectively are

$$\left.\begin{aligned} dA_1 &= l_1 dS \\ dA_2 &= l_2 dS \\ dA_3 &= l_3 dS \end{aligned}\right\} \tag{16}$$

By Poynting's Theorem (eq (16), Chapter III), the energy flowing per second per unit cross section of either of these beams is

$$\mathbf{s} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} \tag{17}$$

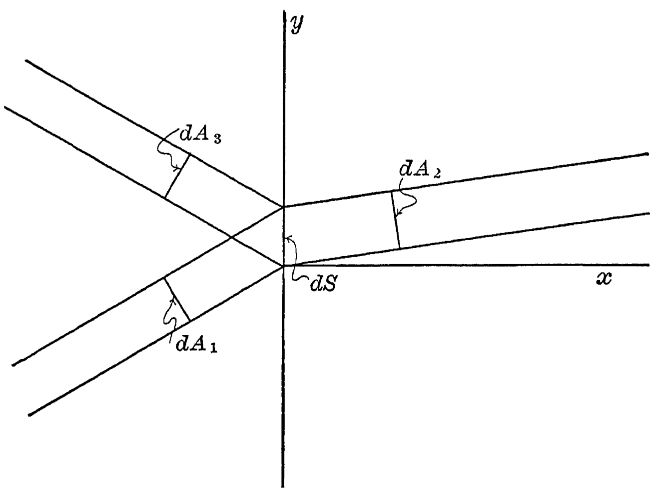

FIG. 2—Relation of areas of cross-section in the several beams.

The energy flowing per second through any area dA perpendicular to the ray is the area times the value of $\mathbf{s}$; *i.e.*,

$$d\mathbf{s} = \frac{cdA}{4\pi}\mathbf{E}\times\mathbf{H}.$$

Substituting the values of the dA's from (16) and the values of of the various $\mathbf{H}$'s in terms of their $\mathbf{E}$'s from (7), we have for the energy per second at dS on the surface between the media, the values

$$\text{Incident Energy per Sec.} = \frac{cdS}{4\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\,l_1 E_{1_0}{}^2 \tag{18}$$

$$\text{Reflected Energy per Sec.} = \frac{cdS}{4\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\,l_1 E_{3_0}{}^2 \tag{19}$$

$$\text{Refracted Energy per Sec.} = \frac{cdS}{4\pi}\sqrt{\frac{\epsilon_2}{\mu_2}}\, l_2 E_{2_0}{}^2 \tag{20}$$

where the subscript ($_0$) indicates that values at the mirror are to be taken; that is, values with $x = 0$.

Calling the ratio of the reflected energy per second to the incident energy per second the *coefficient of reflection*, indicated by r, we have by (18) and (19)

$$r = \frac{E_{3_0}{}^2}{E_{1_0}{}^2} \tag{21}$$

and by the law of the conservation of energy, from (18), (19) and (20), by equating incident energy to reflected plus refracted energy and dividing out a common factor

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\, l_1 E_{1_0}{}^2 = \sqrt{\frac{\epsilon_1}{\mu_1}}\, l_1 E_{3_0}{}^2 + \sqrt{\frac{\epsilon_2}{\mu_2}}\, l_2 E_{2_0}{}^2$$

whence by transposition and division,

$$1 - r = \frac{l_2}{l_1}\frac{E_{2_0}{}^2}{E_{1_0}{}^2}\sqrt{\frac{\epsilon_2}{\epsilon_1}\frac{\mu_1}{\mu_2}} \tag{22}$$

The equations (21) and (22) hold for any orientation of the electric vector in the plane of the incident wave.

It is proposed now to determine the coefficient of reflection in terms of the index of refraction and angle of incidence alone; for *two principal directions* of polarization of the electric wave. This is done in Art. **53** for **E** perpendicular to the plane of incidence, and Art. **54** for **E** parallel to the plane of incidence.

53. Determination of Coefficient of Reflection when E is Perpendicular to the Plane of Incidence.—In this case, since the plane of incidence is the xy-plane, we have the **E** entirely in the z-direction; that is,

$$E = E_z.$$

As before, let us indicate by a subscript ($_0$) the value of E at the reflecting surface.

From the continuity of the tangential component of electric force at the surface, since the whole force is tangential, we have

$$E_{10} + E_{30} = E_{20} \tag{23}$$

Dividing by E_{10} and substituting from (21) and (22), we obtain

$$1 + \sqrt{r} = \sqrt{\frac{l_1}{l_2}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\,(1 - r)} \tag{24}$$

This squared gives, after factoring,

$$(1 + \sqrt{r})^2 = \frac{l_1}{l_2}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\,(1 - \sqrt{r})(1 + \sqrt{r})$$

Dividing out a common factor, we obtain

$$1 + \sqrt{r} = \frac{l_1}{l_2}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}\,(1 - \sqrt{r}) = \frac{l_1\mu_2}{l_2\mu_1}\sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}}\,(1 - \sqrt{r}) \tag{25}$$

Now

$$\sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}} = \frac{v_2}{v_1};$$

so that (25) may be written

$$l_2\mu_1v_1(1 + \sqrt{r}) = l_1\mu_2v_2(1 - \sqrt{r})$$

whence

$$r = \left(\frac{l_1\mu_2v_2 - l_2\mu_1v_1}{l_1\mu_2v_2 + l_2\mu_1v_1}\right)^2 \tag{26}$$

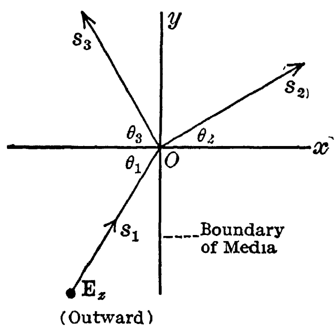

FIG. 3.—**E** perpendicular to the plane of incidence.

In this equation, if Θ_1 and Θ_2 are respectively the angle of incidence and angle of refraction,

$$l_1 = \cos\Theta_1$$

$$l_2 = \cos\Theta_2 = \sqrt{1 - \frac{v_2^2}{v_1^2}\sin^2\Theta_1}, \text{ by (14).}$$

These values substituted in (26) give

$$r = \left(\frac{\mu_2\cos\Theta_1 - \mu_1\sqrt{\dfrac{v_1^2}{v_2^2} - \sin^2\Theta_1}}{\mu_2\cos\Theta_1 + \mu_1\sqrt{\dfrac{v_1^2}{v_2^2} - \sin^2\Theta_1}}\right)^2 \tag{27}$$

Now $\frac{v_1}{v_2} = \frac{n_2}{n_1}$, where n_1 and n_2 are the indices of refraction of incident and refractive media respectively.

In all insulating media $\mu_2 = \mu_1 = 1$, so that (27) may be written

$$r = \left(\frac{\cos \Theta_1 - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2 \Theta_1}}{\cos \Theta_1 + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2 \Theta_1}} \right)^2 \tag{28}$$

Equation (28) *gives the coefficient of reflection r in case the electric force in the incident wave is perpendicular to the plane of incidence and* $\mu_2 = \mu_1 = 1$. *In this equation* n_1 *and* n_2 *are indices of refraction of incident and refractive media respectively and are not to be confused with direction cosines*

54. Determination of the Coefficient of Reflection when E is in the Plane of Incidence.—In this case $E_z = 0$, Fig. 4, and we have for the total electric intensity in each ray

$$E = \sqrt{E_x^2 + E_y^2},$$

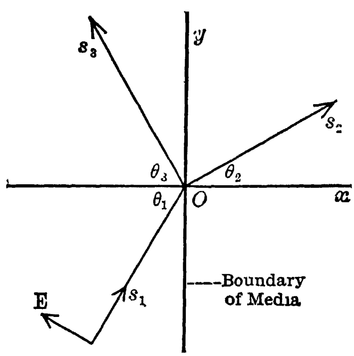

FIG. 4.—**E** in the plane of incidence.

and for the total magnetic intensity

$$H = H_z.$$

The condition of continuity of the tangential component of magnetic intensity at the reflecting surface gives, since the whole magnetic intensity is tangential, the boundary condition

$$H_{10} + H_{30} = H_{20} \tag{29}$$

Expressing now the coefficient of reflection in terms of H_{10}, H_{20}, and H_{30}, by replacing the E's in (21) and (22) by equivalent values in terms of the H's taken from equations (6), we have

$$r = \frac{H_{30}^2}{H_{10}^2}$$

and

$$1 - r = \frac{l_2}{l_1}\frac{H_{20}{}^2}{H_{10}{}^2}\sqrt{\frac{\epsilon_1\mu_2}{\epsilon_2\mu_1}}$$

These values substituted in (29) give

$$1 + \sqrt{r} = \sqrt{\frac{l_1}{l_2}\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}(1 - r)} \tag{30}$$

which is the same as (25) except that the subscripts of ϵ and μ are advanced, and therefore gives on simplification (cf. 26)

$$r = \left(\frac{l_1\mu_1v_1 - l_2\mu_2v_2}{l_1\mu_1v_1 + l_2\mu_2v_2}\right)^2 \tag{31}$$

Replacing l_1 and l_2 by their values in terms of Θ_1, we obtain

$$r = \left\{\frac{\frac{v_1{}^2}{v_2{}^2}\mu_1\cos\Theta_1 - \mu_2\sqrt{\frac{v_1{}^2}{v_2{}^2} - \sin^2\Theta_1}}{\frac{v_1{}^2}{v_2{}^2}\mu_1\cos\Theta_1 + \mu_2\sqrt{\frac{v_1{}^2}{v_2{}^2} - \sin^2\Theta_1}}\right\}^2 \tag{32}$$

or in terms of indices of refraction, when $\mu_1 = \mu_2 = 1$,

$$r = \left\{\frac{\left(\frac{n_2}{n_1}\right)^2\cos\Theta_1 - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\Theta_1}}{\left(\frac{n_2}{n_1}\right)^2\cos\Theta_1 + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\Theta_1}}\right\}^2 \tag{33}$$

Equation (33) *gives the coefficient of reflection r in case the electric force in the incident wave is parallel to the plane of incidence. In this equation n_1 and n_2 are indices of refraction of incident and refractive media respectively.*

55. Transformation of Equations (28) and (33).—By the law of refraction (15), in view of definitions preceding (28), we have

$$\frac{\sin\Theta_1}{\sin\Theta_2} = \frac{n_2}{n_1} \tag{34}$$

where n_1 and n_2 = indices of refraction of incident medium and refractive medium respectively.

From (34)

$$\cos\Theta_2 = \sqrt{1 - \sin^2\Theta_2} = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\Theta_1} \tag{35}$$

Substitution of (35) into (28) gives

$$r = \left\{ \frac{\cos \Theta_1 - \dfrac{n_2}{n_1} \cos \Theta_2}{\cos \Theta_1 + \dfrac{n_2}{n_1} \cos \Theta_2} \right\}^2.$$

Replacing n_2/n_1 by its value from (34), we obtain

$$r = \left\{ \frac{-\sin (\Theta_1 - \Theta_2)}{\sin (\Theta_1 + \Theta_2)} \right\}^2 \quad \left\{ \begin{array}{l} \text{For incident } \mathbf{E} \\ \text{perpendicular to} \\ \text{plane of inci-} \\ \text{dence} \end{array} \right\} \tag{36}$$

Treating (33) in a similar manner, we obtain

$$r = \left\{ \frac{-\tan (\Theta_1 - \Theta_2)}{\tan (\Theta_1 + \Theta_2)} \right\}^2 \quad \left\{ \begin{array}{l} \text{For incident } \mathbf{E} \\ \text{parallel to plane} \\ \text{of incidence} \end{array} \right\} \tag{37}$$

Equations (36) *and* (37) *are known as Fresnel's equations. In these equations r is the ratio obtained by dividing energy per second leaving reflecting surface in reflected beam by energy per second incident on same surface.*

Θ_1 = *angle of incidence.*
Θ_2 = *angle of refraction.*

Equation (36) *is for a plane incident wave with the electric force perpendicular to the plane of incidence. In optics such a wave is said to be polarized in the plane of incidence.*

Equation (37) *is for a plane incident wave with the electric force parallel to the plane of incidence. Such a wave is said to be polarized perpendicular to the plane of incidence.*

The plane of incidence is the plane of the incident ray, the reflected ray and the normal to the reflecting surface.

CHAPTER VII

ELECTRIC WAVES IN AN IMPERFECTLY CONDUCTIVE MEDIUM[1]

56. Wave Equations in a Homogeneous Imperfect Conductor.—It has been shown in Art **35,** Chapter IV, that in a homogeneous medium of conductivity γ, permeability μ, and dielectric constant ϵ, the magnetic and electric intensities satisfy the equations

$$\frac{4\pi\gamma\mu}{c^2}\frac{\partial \mathbf{H}}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla^2 \mathbf{H} \tag{1}$$

and

$$\frac{4\pi\gamma\mu}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \frac{\epsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} - \frac{4\pi}{\epsilon}\operatorname{grad}\rho \tag{2}$$

57. Relaxation Time.—A question now arises as to the value of the intrinsic volume density ρ in such a medium. We can determine this matter by taking the divergence of equation (8), Art. **35,** remembering that the divergence of a curl is zero; we have

$$\frac{4\pi\gamma}{c}\operatorname{div.}\mathbf{E} + \frac{\epsilon}{c}\frac{\partial}{\partial t}(\operatorname{div.}\mathbf{E}) = 0 \tag{3}$$

or replacing div. $\mathbf{E}$ by its value in terms of ρ,

$$\frac{4\pi\gamma}{\epsilon}\rho + \frac{\partial\rho}{\partial t} = 0$$

Integrating this we obtain

$$\left.\begin{aligned} \rho &= \rho_0 e^{-\frac{4\pi\gamma}{\epsilon}t} \\ &= \rho_0 e^{-t/\tau} \end{aligned}\right\} \tag{4}$$

where e is base of natural logarithms and

$$\tau = \frac{\epsilon}{4\pi\gamma} \tag{5}$$

Whence it appears that if ρ has the value ρ_0 at some time reckoned as origin of time, ρ will decrease exponentially with

[1] This chapter is based on ABRAHAM and FÖPPL, "Theorie der Electrizität," Vol. 1, p. 321, 1907.

the time. The process is called *relaxation*, and the time for ρ to fall to one eth of its value is τ, given by (5), and called the *relaxation time* of the material. The relaxation time for any good conductor is so short that it has never been experimentally determined for any metal. Its determination for so poor a conductor as pure water is a matter of extreme difficulty.

58. Steady-state Plane Wave Equation.—Equation (4) shows that after the lapse of a sufficient time, usually very brief, the value of ρ in any conductor is substantially zero, and we may omit the ρ term from (2).

Having thus simplified the equation (2), let us next restrict the wave field to a plane-wave field. Then **E** and **H** will be functions of t and s alone, where s is the perpendicular distance from the origin of coordinates to a plane over which the field is constant at a given time. Then if l, m, and n are the direction cosines of s,

$$s = lx + my + nz \tag{6}$$

is the equation of any such plane, and the quantities $\nabla^2\mathbf{H}$ and $\nabla^2\mathbf{E}$ reduce to $\dfrac{\partial^2\mathbf{H}}{\partial s^2}$ and $\dfrac{\partial^2\mathbf{E}}{\partial s^2}$, so that the wave equations (1) and (2) become

$$\frac{\epsilon\mu}{c^2}\left(\frac{4\pi\gamma}{\epsilon}\frac{\partial\mathbf{H}}{\partial t} + \frac{\partial^2\mathbf{H}}{\partial t^2}\right) = \frac{\partial^2\mathbf{H}}{\partial s^2} \tag{7}$$

and

$$\frac{\epsilon\mu}{c^2}\left(\frac{4\pi\gamma}{\epsilon}\frac{\partial\mathbf{E}}{\partial t} + \frac{\partial^2\mathbf{E}}{\partial t^2}\right) = \frac{\partial^2\mathbf{E}}{\partial s^2} \tag{8}$$

59. Limitation to Solution Harmonic in Time.—Each component of electric intensity and each component of magnetic intensity must satisfy an equation of the form of (8). Let M be the generic designation for E_x, E_y, E_z, H_x, H_y, H_z, then

$$\frac{\epsilon\mu}{c^2}\left(\frac{4\pi\gamma}{\epsilon}\frac{\partial M}{\partial t} + \frac{\partial^2 M}{\partial t^2}\right) = \frac{\partial^2 M}{\partial s^2} \tag{9}$$

This equation is a form of equation that plays a fundamental rôle in telegraphy and telephony and is known as the *telegraph* [illegible] which has been the subject of much theoretical and practical investigation.

We shall content ourselves with a treatment of the equation for the special case in which the solution involves the time har-

monically. M will then be the real part of the quantity that can be written in the form

$$M = e^{j\omega t} F(s) \tag{10}$$

where $F(s)$ is some function of s but not of t.

Designating the second derivative of F with respect to s by F'', and substituting (10) in (9), we have

$$\frac{\epsilon\mu}{c^2}\left(\frac{4\pi\gamma}{\epsilon} j\omega - \omega^2\right)F(s) = F''(s) \tag{11}$$

Since F'' is a complete derivative, (11) is an ordinary differential equation of the second order with constant coefficients, and its solution may be written in the form

$$F(s) = ae^{ks} \tag{12}$$

which substituted in (11) gives

$$k^2 = \frac{\epsilon\mu}{c^2}\left(\frac{4\pi\gamma}{\epsilon}\omega j - \omega^2\right) \tag{13}$$

while a is completely arbitrary.

It is seen that k is a complex quantity. Let us break up k into real and imaginary parts by setting

$$k = -\frac{\omega}{c}(\chi + jn) \tag{14}$$

where χ and n are both real quantities, and χ is *positive* to avoid infinite values of M.

From (14) and (13) we are to determine χ and n.

60. Determination of χ and n.—Substituting (14) in (13) we obtain

$$\chi + jn = \pm\sqrt{\epsilon\mu}\sqrt{\frac{4\pi\gamma}{\epsilon\omega}j - 1} = \pm\sqrt{\mu}\sqrt{2\gamma Tj - \epsilon} \tag{15}$$

where

$$T = 2\pi/\omega = \text{the period.} \tag{16}$$

Squaring and equating real and imaginary parts, we have

$$\chi^2 - n^2 = -\epsilon\mu \tag{17}$$

and

$$2\chi nj = 2\gamma\mu Tj \tag{18}$$

Subtracting (18) from (17) and extracting the square root, we obtain

$$\chi - jn = \pm\sqrt{\mu}\sqrt{-2\gamma Tj - \epsilon} \tag{19}$$

The product of (19) and (15) gives

$$\chi^2 + n^2 = \mu\sqrt{\epsilon^2 + 4\gamma^2 T^2} \tag{20}$$

This compared with (17) gives, by addition and subtraction and by omitting signs inconsistent with the condition that χ and n shall be real and χ shall be positive, the result

$$\chi^2 = \frac{\mu}{2}\{\sqrt{\epsilon^2 + 4\gamma^2 T^2} - \epsilon\} \tag{21}$$

$$n^2 = \frac{\mu}{2}\{\sqrt{\epsilon^2 + 4\gamma^2 T^2} + \epsilon\} \tag{22}$$

M may now be expressed in terms of χ and n by combining (14), (12) and (10), and is

$$M = ae^{-\frac{\omega\chi s}{c}} e^{j\omega(t - ns/c)} \tag{23}$$

where a is an arbitrary constant and is in general a complex quantity. The real part of (23) is also a solution of the given differential equation, and may be written in the form

$$M = Ae^{-\frac{\omega\chi s}{c}} \cos\{\omega(t - ns/c) + \phi\} \tag{24}$$

where A and ϕ are both arbitrary constants. A solution of the form of (24) is the most general harmonic solution of angular velocity ω of the given differential equation (9); for the assumption that the solution is a harmonic function of the time with angular velocity ω reduces the equation to the form of (11), which is an ordinary differential equation of the second order, so that any solution that contains two arbitrary constants is the general solution.

61. Extinction Coefficient, Velocity, and Index of Refraction. Each component of electric and magnetic intensity in a harmonic wave in a homogeneous conductive medium satisfies an equation of the form of (24)—with, however, in general a different value of A and ϕ for each component.

It is seen that the intensities are attenuated as the wave penetrates deeper and deeper into the conductor, and that the attenuation is determined by the factor

$$e^{-\frac{\chi\omega s}{c}}$$

which may be called the *Attenuation Factor*. The quantity χ

is called the *Extinction Coefficient* of the medium for the given frequency of oscillation. The exponential term is expressed in the rather complicated form here given, so that χ shall be a quantity symmetrical in form with n.

A verbal description of the extinction coefficient may be had by substituting

$$\omega = 2\pi/T$$

where T is the period of oscillation, and

$$\lambda_0 = cT,$$

where λ_0 equals the wavelength in vacuo; then the attenuation factor given above becomes

$$e^{-\chi\frac{2\pi s}{\lambda_0}}$$

or

$$e^{-\chi}, \text{ if } s = \lambda_0/2\pi;$$

so the *extinction coefficient* χ *is the logarithmic decrement of amplitude for a traversed distance equal to* $\frac{1}{2\pi}$ *of a vacuum wavelength.*

Returning now to (24), let us see next the significance of n. Apart from the attenuation factor, M is seen to be a function of $t - s/(c/n)$; therefore, the velocity of propagation of a given phase of the wave is

$$v = c/n \tag{25}$$

where c is the velocity of the wave in vacuo. *Hence n is the index of refraction of the conductive medium for the particular frequency.*

By substituting the value of n from (22) in (25), we have for v

$$v = \frac{c}{\sqrt{\frac{\mu}{2}\left\{\sqrt{\epsilon^2 + 4\gamma^2T^2} + \epsilon\right\}}} \tag{26}$$

$$= \frac{c}{\gamma T}\sqrt{\frac{\sqrt{\epsilon^2 + 4\gamma^2T^2} - \epsilon}{2\mu}} \tag{27}$$

$$= \frac{c\chi}{\mu\gamma T} \tag{28}$$

The values of χ, n and v may be simplified for certain special cases by expansion of the radical expressions with neglect of small terms. Examples follow.

62. Special Case of Small Conductivity.—If $\gamma^2 T^2$ is negligible in comparison with $2\epsilon^2$

$$v = \frac{c}{\sqrt{\mu\epsilon}} \tag{29}$$

$$n = \sqrt{\mu\epsilon} \tag{30}$$

$$\chi = \sqrt{\frac{\mu}{\epsilon}}\,\gamma T \tag{31}$$

$$e^{-\frac{\chi\omega s}{c}} = e^{-\frac{2\pi\gamma}{\epsilon v}s} \tag{32}$$

In this special case of low conductivity, the velocity v, the index of refraction n, and the attenuation factor $e^{-\frac{\chi\omega s}{c}}$ are all independent of the frequency of oscillation.

63. Special Case of Large Conductivity.—If, on the other hand, the conductivity is so large in comparison with the dielectric constant that ϵ is negligible in comparison with $4\gamma T$,

$$v = \frac{c}{\sqrt{\mu\gamma T}} \tag{33}$$

$$n = \sqrt{\mu\gamma T} \tag{34}$$

$$\chi = \sqrt{\mu\gamma T} \tag{35}$$

$$e^{-\frac{\chi\omega s}{c}} = e^{-\frac{2\pi}{c}\sqrt{\frac{\mu\gamma}{T}}s} \tag{36}$$

In this special case the velocity, index of refraction, and attenuation factor all involve the square root of the period of oscillation.

64. Relation of H to E.—Each component of **E** can be expressed in the form of (23), where only the real part is to be taken. The y and z-components are

$$E_y = a_y e^{-\frac{\chi\omega s}{c}} e^{j\omega(t - ns/c)}$$

$$E_z = a_z e^{-\frac{\chi\omega s}{c}} e^{j\omega(t - ns/c)},$$

in which

$$s = lx + my + nz.$$

(The direction cosine n is not to be confused with the index of refraction n.)

Now by Maxwell's Equation (B), Chapter III, taking the x-component

$$-\frac{\mu}{c}\frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$= \left(-\frac{\omega\chi}{c} - \frac{\omega n}{c}j\right)\left(mE_z - nE_y\right)$$

Integrating with respect to t, we obtain

$$-\frac{\mu}{c}H_x = \frac{-\frac{\omega}{c}(\chi + nj)}{j\omega}(mE_z - nE_y)$$
$$H_x = \frac{1}{\mu}(-\chi j + n)(mE_z - nE_y)$$
$$= \frac{1}{\mu}\sqrt{n^2 + \chi^2}\,e^{-j\tan^{-1}\left(\frac{\chi}{n}\right)}\{mE_z - nE_y\} \quad (37)$$

The factor $e^{-j\tan^{-1}\left(\frac{\chi}{n}\right)}$ indicates that the real part of (37) may be obtained by taking the real parts of E_z and E_y and retarding their phase angles by $\tan^{-1}\left(\frac{\chi}{n}\right)$. If we indicate such a retardation of phase by the engineering symbol $\overline{\backslash \tan^{-1}\left(\frac{\chi}{n}\right)}$, we have the real equation

$$H_x = \frac{1}{\mu}\sqrt{n^2 + \chi^2}\left\{(mE_z - nE_y)\overline{\backslash\tan^{-1}\left(\tfrac{\chi}{n}\right)}\right\} \quad (38)$$

The expression in braces is seen to be the x-component of the vector product

$$\mathbf{U}_s \times \mathbf{E}\overline{\backslash\tan^{-1}\left(\tfrac{\chi}{n}\right)}$$

where $\mathbf{U}_s$ is a unit vector in the s-direction.

There are similar components for H_y and H_z; so that the total vector $\mathbf{H}$ may be written

$$\mathbf{H} = \frac{1}{\mu}\sqrt{n^2 + \chi^2}\,\mathbf{U}_s \times \mathbf{E}\overline{\backslash\tan^{-1}\left(\tfrac{\chi}{n}\right)} \quad (39)$$

This equation means that $\mathbf{H}$ *is the positive perpendicular to* s *and to* $\mathbf{E}$, *that the magnitude of* $\mathbf{H}$ *is* $\frac{1}{\mu}\sqrt{n^2 + \chi^2}$ *times the magnitude of* $\mathbf{E}$, *and that* $\mathbf{H}$ **lags** *behind* $\mathbf{E}$ *in phase by the angle whose tangent is* χ/n.

Written trigonometrically, with the aid of (24), if the magnitude of the resultant electric intensity is

$$E = Ae^{-\frac{\chi\omega s}{c}}\cos\left\{\omega(t - ns/c) + \phi\right\} \quad (40)$$

then

$$H = \frac{1}{\mu}\sqrt{\chi^2 + n^2}\,Ae^{-\frac{\chi\omega s}{c}}\cos\left\{\omega(t - ns/c) + \phi - \tan^{-1}\left(\frac{\chi}{n}\right)\right\} \quad (41)$$

65. Poynting's Vector. Transmission and Absorption of Energy.—We shall next determine the amount of energy flowing per unit cross section per second in the direction of s. The general form of Poynting's vector is

$$\mathbf{s} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$$

which gives for the problem under consideration

$$\mathbf{s} = \frac{c}{4\pi} A^2 \frac{\sqrt{n^2 + \chi^2}}{\mu} e^{-\frac{2\chi\omega s}{c}} \cos\alpha \cos\left\{\alpha - \tan^{-1}\left(\frac{\chi}{n}\right)\right\} \mathbf{U}_S$$

where

$$\alpha = \omega(t - ns/c) + \phi$$

$$\mathbf{U}_S = \text{a unit vector in the direction } s.$$

Expanding the second cosine factor, and taking the time average, indicated by $\bar{\mathbf{s}}$, we obtain

$$\bar{\mathbf{s}} = \frac{c}{4\pi}\frac{n}{\mu}\frac{A^2}{2} e^{-\frac{2\chi\omega s}{c}} \mathbf{U}_s \tag{42}$$

$$= \frac{c}{4\pi}\frac{n}{\mu}\frac{A_s^2}{2} \mathbf{U}_s \tag{43}$$

where

$$A_s = \text{amplitude of } E \text{ at } s.$$

Equation (42) or (43) gives the average rate of flow of energy per second per unit area within the conductor.

It is easy to obtain from this expression (42) the *average rate at which energy is absorbed in the conductor.* The absorbed energy per unit volume per second indicated by $\bar{P}$ is the decrease of $\bar{\mathbf{s}}$ per unit distance,

$$\left.\begin{aligned} \bar{P} &= -\frac{\partial\bar{\mathbf{s}}}{\partial s} = \frac{c}{4\pi}\frac{n}{\mu}\frac{A^2}{2}\frac{2\chi\omega}{c} e^{-\frac{2\chi\omega s}{c}} \\ &= \gamma\frac{\left(Ae^{-\frac{\chi\omega s}{c}}\right)^2}{2} \\ &= \gamma\frac{A_s^2}{2} \end{aligned}\right\} \tag{44}$$

where, again,

$$A_s = \text{amplitude of } E \text{ at } s.$$

The same result may be obtained by taking the time average of electromotive force per unit length times current-density.

Equation (43) gives the average power transmitted per unit area and equation (44) gives the average power absorbed per unit volume.

66. The Reflection of a Harmonic Plane Polarized Wave from a Plane Imperfectly Conductive Surface at Normal Incidence. In Chapter V the reflection from a perfect conductor has been considered. It is proposed to investigate now the reflection at normal incidence of a plane harmonic wave from a surface of a conductor of any conductivity γ, dielectric constant ϵ, and permeability μ.

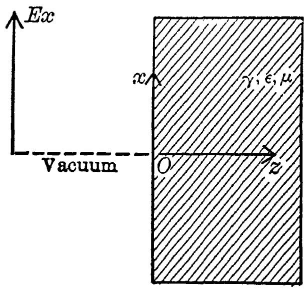

FIG. 1.—Illustrating a plane wave incident normally on the surface of a medium of any conductivity γ, dielectric constant ϵ and permeability μ

Let the surface of the conductor be through the origin of coordinates and in the xy-plane, Fig. 1, and let the x-axis be in the direction of the electric intensity. Let a plane electric wave traveling in a vacuum in the z-direction fall upon the conductive surface of which the conductivity, dielectric constant and permeability are respectively γ, ϵ, and μ.

Indicating by subscript ($_1$) the direct wave; by ($_2$) the transmitted wave, and by ($_3$) the reflected wave, we have

In the Direct Wave (incident)

$$E_{1x} = A_1 \cos \omega(t - z/c) \tag{45}$$

$$H_1 = A_1 \cos \omega(t - z/c) \tag{46}$$

and Poynting's vector

$$\mathbf{s}_1 = \frac{c}{4\pi} A_1{}^2 \cos^2 \omega(t - z/c)\, \mathbf{U}_z,$$

of which the time average is

$$\bar{\mathbf{s}}_1 = \frac{c}{8\pi} A_1{}^2\, \mathbf{U}_z \tag{47}$$

In the Reflected Wave

$$E_{3_x} = \quad A_3 \cos \{\omega(t + z/c) + \phi_3\} \tag{48}$$

$$H_{3_y} = - A_3 \cos \{\omega(t + z/c) + \phi_3\} \tag{49}$$

$$\mathbf{s}_3 = -\frac{c}{8\pi} A_3{}^2\, \mathbf{U}_z \tag{50}$$

In the Transmitted Wave

$$E_{2_x} = A_2 e^{-\frac{\chi\omega z}{c}} \cos\{\omega(t - nz/c) + \phi_2\} \tag{51}$$

$$H_{2_y} = \frac{1}{\mu}\sqrt{n^2+\chi^2}\;A_2 e^{-\frac{\chi\omega z}{c}} \cos\left\{\omega(t - nz/c) + \phi_2 - \tan^{-1}\left(\frac{\chi}{n}\right)\right\} \tag{52}$$

$$\bar{\mathrm{s}}_2 = \frac{c}{8\pi}\frac{n}{\mu}A_2{}^2\, e^{-\frac{2\chi\omega z}{c}}\,\mathbf{U}_z \tag{53}$$

If now we consider a unit area of the reflecting surface of the conductor, the law of the conservation of energy, which applies to the instantaneous values and, therefore, to the time averages, gives

$$\bar{\mathrm{s}}_{1_0} + \bar{\mathrm{s}}_{3_0} = \bar{\mathrm{s}}_{2_0} \tag{54}$$

where the subscripts ($_0$) indicates that the values at the surface ($z = 0$) are meant. Whence by (47), (50), and (53)

$$A_1{}^2 - A_3{}^2 = \frac{n}{\mu}A_2{}^2 \tag{55}$$

The coefficient of reflection r is defined as the numerical ratio of the average energy reflected per second to the average energy incident per second; therefore,

$$r = \frac{\bar{\mathrm{s}}_{3_0}}{\bar{\mathrm{s}}_{1_0}} = \frac{A_3{}^2}{A_1{}^2} \tag{56}$$

and from (55), by division by $A_1{}^2$,

$$1 - r = \frac{n}{\mu}\frac{A_2{}^2}{A_1{}^2} \tag{57}$$

For the purpose of determining r numerically, we need next the fact that *the tangential components of E and H are continuous* at the surface between the media. This gives

$$A_1 \cos\omega t + A_3 \cos(\omega t + \phi_3) = A_2 \cos(\omega t + \phi_2) \tag{58}$$

and

$$A_1 \cos\omega t - A_3 \cos(\omega t + \phi_3) = A_2\frac{\sqrt{n^2+\chi^2}}{\mu}\cos\left\{\omega t + \phi_2 - \tan^{-1}\left(\frac{\chi}{n}\right)\right\}$$
$$= A_2\left\{\frac{n}{\mu}\cos(\omega t + \phi_2) + \frac{\chi}{\mu}\sin(\omega t + \phi_2)\right\} \tag{59}$$

Setting $\omega t = \pi/2$ and taking the sum of (58) and (59), the left-hand side sums up to zero, and we have

$$0 = \left(1 + \frac{n}{\mu}\right)\sin\phi_2 - \frac{\chi}{\mu}\cos\phi_2$$

whence

$$\tan\phi_2 = \frac{\chi}{\mu + n} \tag{60}$$

Now taking the sum of (58) and (59) and making $\omega t = 0$, we have

$$2A_1 = A_2\left\{\frac{\mu + n}{\mu}\cos\phi_2 + \frac{\chi}{\mu}\sin\phi_2\right\};$$

and by (60) this reduces to

$$2A_1 = A_2\frac{\sqrt{(\mu + n)^2 + \chi^2}}{\mu}$$

Therefore,

$$\left(\frac{A_2}{A_1}\right)^2 = \frac{4\mu^2}{(\mu + n)^2 + \chi^2}$$

and by (57)

$$1 - r = \frac{n}{\mu}\left(\frac{A_2}{A_1}\right)^2 = \frac{4n\mu}{(\mu + n)^2 + \chi^2}$$

$$r = \frac{(\mu - n)^2 + \chi^2}{(\mu + n)^2 + \chi^2} \tag{61}$$

where, by (21) and (22),

$$\chi^2 = \frac{\mu}{2}\{\sqrt{\epsilon^2 + 4\gamma^2T^2} - \epsilon\} \tag{62}$$

$$n^2 = \frac{\mu}{2}\{\sqrt{\epsilon^2 + 4\gamma^2T^2} + \epsilon\} \tag{63}$$

Equation (61) gives the coefficient of reflection r at normal incidence of a harmonic electric wave of period T from the plane surface of a homogeneous body of conductivity γ, dielectric constant ϵ, and permeability μ in contact with a vacuum.

67. Special Case for Conductivity Zero.—The equation (61) is true in general for normal incidence whatever the value of the conductivity. If $\gamma = 0$, $\chi = 0$, and with $\mu = 1$, this reduces to

$$r = \frac{(1 - n)^2}{(1 + n)^2}$$

which is the equation to which (28) and (33), Chapter VI, derived for vitreous reflection, also reduce when the incidence is normal,

i e., $\Theta_1 = 0$ and the first medium is a vacuum. (N. B. The quantity n of (63) reduces in this case to the familiar index of refraction $n = \sqrt{\mu\epsilon}$)

68. Special Case of a Good Conductor.—In this case if we assume ϵ negligible in comparison with $4\gamma T$, we have by (34) and (35)

$$n = \chi = \sqrt{\mu\gamma T},$$

whence the coefficient of reflection r *of* (61) becomes

$$\begin{aligned} r &= \frac{\mu^2 - 2\mu\sqrt{\mu\gamma T} + 2\mu\gamma T}{\mu^2 + 2\mu\sqrt{\mu\gamma T} + 2\mu\gamma T} \\ &= \frac{2\gamma T - 2\sqrt{\mu\gamma T} + \mu}{2\gamma T + 2\sqrt{\mu\gamma T} + \mu} \\ &= 1 - \frac{2\sqrt{\mu}}{\sqrt{\gamma T}} + \frac{2\mu}{\gamma T} + \ldots \text{ (approx.)} \end{aligned} \tag{64}$$

This law has been tested for the reflection of long heat waves from metals in some experiments by Hagen and Rubens[1] and has been found to agree with the facts within the limits of the errors of measurement for the metals tested, except bismuth.

69. Phase Changes at Reflection at Normal Incidence.—In equation (60) we have obtained the value

$$\phi_{2_E} = \tan^{-1}\frac{\chi}{\mu + n} \tag{65}$$

This angle ϕ_2 is the angle of advance of the phase of the *transmitted electric intensity* over the phase of the incident electric intensity.

The corresponding angle for the *transmitted magnetic intensity* is

$$\phi_{2_H} = \phi_{2_E} - \tan^{-1}\left(\frac{\chi}{n}\right).$$

To obtain the *phase angle of the reflected wave*, we may use equation (58), which for $\omega t = \pi/2$ becomes

$$A_3 \sin\phi_3 = A_2 \sin\phi_2.$$

In view of (56) and (57) this may be written

$$\sin\phi_3 = \sqrt{\frac{\mu}{n}\frac{1-r}{r}}\sin\phi_2,$$

[1] E. HAGEN and H. RUBENS, *Ann. der Physik.* (4), Vol. II., p. 873, 1903.

which by (65) gives, after proper transformations,

$$\tan \phi_{3_E} = \tan \phi_{3_H} = \frac{2\mu\chi}{\chi^2 + n^2 - \mu^2} \qquad (66)$$

This angle ϕ_3 is the *angle of advance of phase of the electric or magnetic intensity* of the reflected beam over the incident beam, by reflection at normal incidence.

CHAPTER VIII

ELECTRIC WAVES DUE TO AN OSCILLATING DOUBLET

70. Doublet Consisting of an Electron Oscillating in a Positive Atom.—One conception of an oscillating doublet based on the Thomson Atom[1] is illustrated in Fig. 1. This system is supposed to consist of a large positively charged and practically immovable positive sphere of uniform charge density, within which a small negatively charged body (an electron) is oscillating about its position of equilibrium at the center of the sphere. Let the distance of the electron from the center of the atom be p. Let the charge of the electron be $-e$, and the charge of the positive sphere be $+e$. If every element of the sphere attracts the electron with a force inversely proportional to the square of the distance from the element to the electron, the total force on the electron will be proportional to the distance p and proportional to e^2, and will be in the line joining the electron with the center of the sphere; that is,

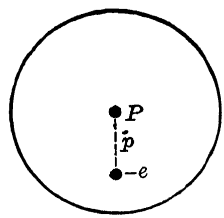

FIG. 1.—A doublet consisting of a negative electron, $-e$, capable of oscillating within a uniformly charged solid sphere.

$$\text{Restoring force} = A = Ke^2p$$

The static energy of the system will then be

$$W_s = \int_0^p A\,dp = \frac{1}{2}Ke^2p^2 = \frac{1}{2}K\{f(t)\}^2 \tag{1}$$

where

$$f(t) = ep = \text{moment of the doublet} \tag{2}$$

K = restoring force per unit distance per unit charge.

The kinetic energy of the system is

$$W_k = \frac{1}{2}m\dot{p}^2 = \frac{1}{2}M\dot{f}^2 \tag{3}$$

where

$$M = \frac{m}{e^2} \tag{4}$$

[1] SIR J. J. THOMSON, "The Corpuscular Theory of Matter," London, 1907.

In modern electron theory the mass m and therefore the quantity M in this expression for the kinetic energy is a constant only provided the velocity of the electron is small in comparison with the velocity of light. We shall need this assumption later for other reasons. The total energy of the system is

$$U = \frac{1}{2} Kf^2 + \frac{1}{2} M\dot{f}^2 \tag{5}$$

71. Alternative Conception of Doublet Leading to Equivalent Results.—An alternative type of oscillator leading to the same form of energy equation is illustrated in Fig. 2. Two bodies A and B of large mutual capacity are connected by a short wire of zero resistance, and electric currents are supposed to flow between A and B giving them at any time equal and opposite charges q. The capacities of the bodies A and B are supposed to be so large that the capacity of the connecting wire may be neglected. Then the same current i will flow throughout the length of the connecting wire, and $i = \dot{q}$. If C is the mutual capacity of A and B, the static energy of the system will be

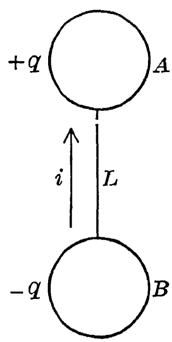

FIG. 2—Dumb-bell doublet

$$W_s = \frac{1}{2} \frac{q^2}{C}.$$

The energy in the inductance L, which is the inductance of the connecting wire, is

$$W_L = \frac{1}{2} Li^2$$
$$= \frac{1}{2} L\dot{q}^2$$

Whence the total energy of the system is

$$U = \frac{1}{2} \frac{q^2}{C} + \frac{1}{2} L\dot{q}^2 \tag{6}$$

If p is the distance apart of A and B, and we write the moment of this system

$$f(t) = qp,$$

we have

$$U = \frac{1}{p^2} \left\{ \frac{1}{2} \frac{f^2}{C} + \frac{1}{2} L\dot{f}^2 \right\} \tag{7}$$

which is of the same form as (5)

In this alternative type of doublet, the distance between A and B must be small in comparison with the wavelength of the free oscillation of the system, so that the distributed capacity in the lead wire L may be neglected.

72. Oscillations with Constant Energy.—If, with the first type of doublet, we assume the energy U constant we shall have

$$\dot{U} = 0 = Kf\dot{f} + M\dot{f}\ddot{f},$$

which divided by $\dot{f}$ and integrated gives

$$f = A_1 \cos(\omega_0 t + \phi) \tag{8}$$

where A_1 and ϕ are arbitrary constants and

$$\omega_0 = \sqrt{\frac{K}{M}} \tag{9}$$

A similar treatment of the second type of doublet gives the same value of f, but with

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{10}$$

The oscillation in either case would go on undiminished with constant amplitude and frequency, if the system did not radiate or receive any energy. We shall next show how to calculate the energy radiated as electromagnetic waves from an oscillator of these types. But we shall arrive at the result only by an indirect and somewhat tedious process.

73. Treatment of a Polarized Spherical Wave.—In this we shall follow the method of Hertz.[1] Without at present entering into a consideration of the source of the waves, let us consider an electromagnetic field in which the component of magnetic intensity in the z-direction is zero; that is

$$H_z = 0.$$

We shall assume that the medium is homogeneous everywhere except near the origin of coördinates, where there will be located an oscillator of, as yet, an undefined character.

In any part of the medium, whether homogeneous or not, the z-component of Maxwell's Equation (B) gives

$$0 = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \tag{11}$$

[1] HERTZ, "Electric Waves," translated by D E Jones, Macmillan and Co., 1893. See also Planck, "Wärmestrahlung," Barth, p 100, 1906.

It follows that for the two components E_x and E_y a scalar function V exists such that

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y} \tag{12}$$

as may be proved by a cross differentiation that leads to (11).

Let us next assume that outside of the source the medium has no intrinsic charge, so

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0,$$

from which by substitution from (12),

$$\frac{\partial E_z}{\partial z} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \tag{13}$$

An examination of (13) suggests making V the z-derivative of some function F so that the equation (13) can be integrated. Let

$$V = -\frac{\partial F}{\partial z} \tag{14}$$

Then from (12), (13) and (14) we have

$$\left.\begin{aligned} E_x &= \frac{\partial}{\partial x}\frac{\partial F}{\partial z} \\ E_y &= \frac{\partial}{\partial y}\frac{\partial F}{\partial z} \\ E_z &= -\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = \frac{\partial^2 F}{\partial z^2} - \nabla^2 F \end{aligned}\right\} \tag{15}$$

Let us now write down two of Maxwell's Equations (A), Chapter III, which are, for $H_z = 0$, and for $u_x = u_y = 0$

$$\begin{aligned} \frac{\epsilon}{c}\frac{\partial E_x}{\partial t} &= -\frac{\partial H_y}{\partial z} \\ \frac{\epsilon}{c}\frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial z} \end{aligned} \tag{16}$$

Substituting from (15) into (16) and integrating we obtain

$$\left.\begin{aligned} H_x &= \frac{\epsilon}{c}\frac{\partial}{\partial t}\frac{\partial F}{\partial y} \\ H_y &= -\frac{\epsilon}{c}\frac{\partial}{\partial t}\frac{\partial F}{\partial x} \\ H_z &= 0 \end{aligned}\right\} \tag{17}$$

Equations (15) and (17) show that, without any assumption other than that $\rho = H_z = u_x = u_y = 0$, we have been able to express all of the components of electric and magnetic intensity in terms of the derivatives of F, which is a scalar function of x, y, z, and t; so far as we have seen up to the present F may be any such function.

F is, however, not completely arbitrary, for the x-component of Maxwell's Equation (B), Chapter III, is

$$-\frac{\mu}{c}\frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$= -\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial y} - \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial y} - \frac{\partial^2}{\partial z^2}\frac{\partial F}{\partial y} \qquad \text{by (15)}$$

$$= -\nabla^2\left(\frac{\partial F}{\partial y}\right)$$

Replacing the left-hand side of this equation by its value from (17), we have

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial F}{\partial y} = \nabla^2\left(\frac{\partial F}{\partial y}\right),$$

which integrated with respect to y gives

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 F}{\partial t^2} = \nabla^2 F \tag{18}$$

In performing this integration we have neglected the arbitrary functions independent of y, which the integration gives as additive terms to (18). These may be added *ad lib.*, and when added give an equation for F less restrictive than (18). If we restrict F to (18) we shall have it at least sufficiently restricted.

We may say then that given any scalar function F satisfying equation (18), and performing on it the operations indicated in (15) and (17), we shall obtain for points outside of the region of intrinsic charge a set of possible values of electric and magnetic intensities that will make $H_z = u_x = u_y = 0$.

We shall now put a further restriction of F; namely, we shall assume F a function of only t and the distance r from the origin of coördinates:

$$F = F(r, t) \tag{19}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}$$

Preparatory to substituting (19) in (18), we have

$$\frac{\partial F}{\partial x} = \frac{x}{r}\frac{\partial F}{\partial r}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{r}\frac{\partial F}{\partial r} - \frac{x^2}{r^3}\frac{\partial F}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2 F}{\partial r^2}$$

with similar terms for the y and z-derivatives, giving

$$\nabla^2 F = \frac{2}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rF\right) \tag{20}$$

This result substituted in (18) gives

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\left(rF\right) = \frac{\partial^2}{\partial r^2}\left(rF\right) \tag{21}$$

The integration of this equation as in § **40** gives

$$\left.\begin{aligned} rF &= f\left(t - \frac{r}{v}\right) + g\left(t + \frac{r}{v}\right) \\ F &= \frac{1}{r}f\left(t - \frac{r}{v}\right) + \frac{1}{r}g\left(t + \frac{r}{v}\right) \end{aligned}\right\} \tag{22}$$

where

$$v = \frac{c}{\sqrt{\mu\epsilon}}$$

Let us confine our attention to the value of F given by the first of these terms, the f-term, which is a spherical wave of F traveling in the positive direction of r with the velocity v.

In differentiating (22) for substitution in the equations of $E_x, \ldots H_x, \ldots$, let us call

$$\frac{\partial}{\partial t}f\left(t - \frac{r}{v}\right) = \dot{f}$$

and

$$\frac{\partial f(t - r/v)}{\partial(t - r/v)} = f'$$

It is to be noted that

$$\dot{f} = f'\frac{\partial}{\partial t}\left(t - \frac{r}{v}\right) = f'$$

So that we can express all of the derivatives of f in terms of $\dot{f}$; for example

$$
\left.
\begin{aligned}
\frac{\partial f}{\partial r} &= -\frac{f'}{v} = -\frac{\dot{f}}{v} \\
\frac{\partial F}{\partial z} &= -\frac{z}{r^3}f - \frac{z}{r^2 v}\dot{f} \\
\frac{\partial}{\partial x}\frac{\partial F}{\partial z} &= \frac{xz}{r^3}\left\{\frac{3f}{r^2} + \frac{3\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} \\
\frac{\partial^2 F}{\partial z^2} &= \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)f + \left(\frac{3z^2}{r^4 v} - \frac{1}{r^2 v}\right)\dot{f} + \frac{z^2}{r^3 v^2}\ddot{f} \\
\nabla^2 F &= \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = \frac{1}{rv^2}\ddot{f}
\end{aligned}
\right\} \quad (23)
$$

Substituting these values in (15) and (17), we obtain

$$
\left.
\begin{aligned}
E_x &= \frac{xz}{r^3}\left\{\frac{3f}{r^2} + \frac{3\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} \\
E_y &= \frac{yz}{r^3}\left\{\frac{3f}{r^2} + \frac{3\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} \\
E_z &= \frac{2}{r^2}\left\{\frac{f}{r} + \frac{\dot{f}}{v}\right\} - \frac{x^2 + y^2}{r^3}\left\{\frac{3f}{r^2} + \frac{3\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\}
\end{aligned}
\right\} \quad (24)
$$

$$
\left.
\begin{aligned}
H_x &= -\frac{\epsilon}{c}\frac{y}{r^2}\left\{\frac{\dot{f}}{r} + \frac{\ddot{f}}{v}\right\} \\
H_y &= \frac{\epsilon}{c}\frac{x}{r^2}\left\{\frac{\dot{f}}{r} + \frac{\ddot{f}}{v}\right\} \\
H_z &= 0
\end{aligned}
\right\} \quad (25)
$$

Equations (24) and (25) give the values of the electric and magnetic intensities at the point x, y, z in terms of the coördinates of the point and in terms of f and its time derivatives.

It is to be noted that

$$xE_x + yE_y + zE_z \neq 0 \quad (26)$$

$$xH_x + yH_y + zH_z = 0 \quad (27)$$

$$E_xH_x + E_yH_y + E_zH_z = 0 \quad (28)$$

Whence **H** is perpendicular to r in Fig. 3, and (since $H_z = 0$) to z. Hence **H** is tangent to the sphere and also tangent to the sectional circle normal to the z-axis.

H is perpendicular to **E**, but
E is not perpendicular to r, and hence is not tangent to the spherical surface.

Let us transform our equations to spherical coordinates, Fig. 3,

and let ϕ = the longitude of the point x, y, z,
Θ = its colatitude,
r = its distance from the center
ρ = the radius of the small circle in plane perpendicular to z.

Then

$$r = \sqrt{x^2 + y^2 + z^2} \tag{29}$$

$$\rho = \sqrt{x^2 + y^2} \tag{30}$$

Let us now determine the components of **H** and **E** along ϕ, Θ and r in the direction of the increasing value of these coordinates. These components will be designated by the use of ϕ, Θ and r as subscripts.

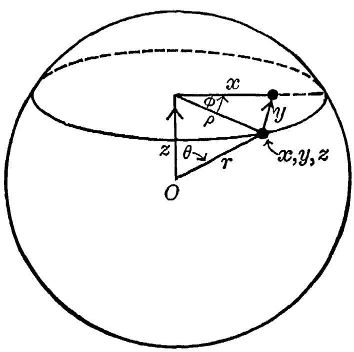

FIG. 3.—Spherical coordinates.

$$\begin{aligned} H_\phi &= H_y \cos\phi - H_x \sin\phi \\ &= H_y \frac{x}{\rho} - H_x \frac{y}{\rho} \\ &= \frac{\epsilon}{c}\frac{\rho}{r^2}\left\{\frac{\dot f}{r} + \frac{\ddot f}{v}\right\} \\ &= \frac{\epsilon}{c}\frac{\sin\Theta}{r}\left\{\frac{\dot f}{r} + \frac{\ddot f}{v}\right\} \end{aligned} \tag{31}$$

The Θ and r-components of **H** are zero.

$$\begin{aligned} E_r &= E_x \frac{x}{r} + E_y \frac{y}{r} + E_z \frac{z}{r} \\ &= \frac{2z}{r^3}\left\{\frac{f}{r} + \frac{f}{v}\right\} \\ &= \frac{2\cos\Theta}{r^2}\left\{\frac{f}{r} + \frac{\dot f}{v}\right\} \end{aligned} \tag{32}$$

$$
\begin{aligned}
E_\Theta &= E_\rho \cos\Theta - E_z \sin\Theta \\
&= \left(E_x \frac{x}{\rho} + E_y \frac{y}{\rho}\right) \cos\Theta - E_z \sin\Theta \\
&= (E_x x + E_y y)\frac{z}{r\rho} - E_z \frac{\rho}{r}, \\
&= \left\{\frac{x^2z^2}{r^4\rho} + \frac{y^2z^2}{r^4\rho} + \frac{(x^2+y^2)\rho}{r^4}\right\}\left\{\frac{3f}{r^2} + \frac{3\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} - \frac{2\rho}{r^3}\left\{\frac{f}{r} + \frac{\dot{f}}{v}\right\} \\
&= \frac{\rho}{r^2}\left\{\frac{f}{r^2} + \frac{\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} \\
&= \frac{\sin\Theta}{r}\left\{\frac{f}{r^2} + \frac{\dot{f}}{rv} + \frac{\ddot{f}}{v^2}\right\} \qquad (33)
\end{aligned}
$$

The ϕ-component of **E** is zero.

Equations (31), (32), *and* (33) *give the values of the components of* **H** *and* **E** *along the spherical coordinates. It is seen that* **H** *is in the direction of the parallels of latitude, and that* **E** *has a component in the direction of the radius* r, *and another component in the direction of the meridianal line.*

Let us now investigate the electric and magnetic field in the neighborhood of the origin, in order to determine the character of the oscillator that could give rise to the field under consideration.

74. Proof that the Field Here Given is the Field Due to a Doublet at r = 0.—In the equations for the components of **H** and **E**, let us investigate the field at distances r from the origin, and suppose that r is so small that

$$\frac{[\dot{f}]}{v} << \frac{[f]}{r} \qquad (34)$$

where the symbol $<<$ means "is negligible in comparison with."

The meaning of this assumption becomes clear when we consider f to be a periodic function of the time with angular velocity ω; then the amplitude of $\dot{f}$ is ω times the amplitude of f. Thus (34) becomes

$$\frac{\omega}{v} << \frac{1}{r}$$

or

$$\frac{2\pi}{vT} << \frac{1}{r}$$

or

$$r << \lambda/2\pi \qquad (35)$$

Under these conditions, the fourth and fifth equations of (23)

show that $\Delta^2 F$ is negligible in comparison[1] with $\frac{\partial^2 F}{\partial z^2}$, and that the E_z of (24) reduces to $\frac{\partial^2 F}{\partial z^2}$, so that by (24) E_x, E_y, and E_z are respectively the x, y, and z-derivatives of the same quantity $\frac{\partial F}{\partial z}$; whence we see that the electric force at this position near the origin of coordinates has an ordinary static potential function

$$\Psi = -\frac{\partial F}{\partial z} \tag{36}$$

and by the second equation of (23), neglecting small terms, we obtain

$$\Psi = \frac{z}{r^3} f \tag{37}$$

We shall now show that this is the potential due to a doublet at the origin with the moment ϵf, provided the square of the length of the doublet is negligible in comparison with $4r^2$.

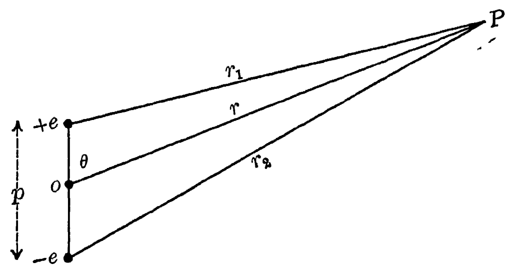

FIG. 4.

In Fig. 4 suppose two charges e and $-e$ separated by a distance p, lying along the direction of the z-axis, and suppose that the point P is distant r from the origin of coördinates midway between the charges, then the electrostatic potential at P is

$$\begin{aligned}\Psi_1 &= \frac{e}{\epsilon r_1} - \frac{e}{\epsilon r_2} \\ &= \frac{e}{\epsilon\left(r - \frac{p\cos\Theta}{2}\right)} - \frac{e}{\epsilon\left(r + \frac{p\cos\Theta}{2}\right)} \\ &= \frac{pe\cos\Theta}{\epsilon\left(r^2 - \frac{p^2\cos^2\Theta}{4}\right)}.\end{aligned}$$

[1] Unless $\frac{\partial^2 F}{\partial z^2}$ becomes small, as it does for certain relations of z to r. In that case the whole force component E_z becomes negligible in comparison with E_x or E_y.

Let us now impose the condition that

$$p^2 << 4r^2 \tag{38}$$

then, since $\cos \Theta = z/r$ (compare Fig. 3),

$$\Psi_1 = \frac{z}{r^3}\frac{pe}{\epsilon} \tag{39}$$

Comparing (37) with (39) it is seen that if $\epsilon = 1$, the potential Ψ of the electromagnetic field at points near the origin of coördinates is the potential of a doublet Ψ of moment (*cf.* (2))

$$pe = f \tag{40}$$

If, on the other hand, the dielectric constant of the medium is different from unity, the moment of the doublet must be

$$pe = \epsilon f \tag{41}$$

in order to have a field continuous with the dynamic electromagnetic field at points near the oscillator.

The conclusion is that the electromagnetic field given by the dynamic equations (24) and (25), or the alternative polar expressions (31), (32) and (33), satisfies the boundary condition imposed by a doublet of moment ϵf at the origin; but this doublet must be so short that the square of its length

$$p^2 << 4r_1^2$$

where, by (35),

$$r_1 << \lambda/2\pi$$

To cause an error of less than one per cent. in the computations,

$$p \overline{\lesssim} .002\,\lambda/2\pi \overline{\lesssim} \lambda/3000, \text{ approx.}$$

This means in the case of a doublet of the type described in Art. **70** that the velocity of the moving electron must be not greater than 1/1500 of the velocity of light. In the alternative type of doublet described in Art. **71** the length between the capacities A and B, Fig. 25, must be not greater than 1/1500 of the radiated wavelength.

We may now continue with the problem under these limitations.

75. Electric and Magnetic Intensities at Great Distance from the Oscillator.—Let us now consider the electric and magnetic

intensities at a point distant r from the oscillator, where r is so great in comparison with the wavelength that

$$[\dot{f}/r] << [\ddot{f}/v],$$

and *a fortiori*

$$[f/r^2] << [\dot{f}/v^2]$$

This means for f a harmonic or nearly harmonic function of the time that

$$r >> \lambda/2\pi.$$

Under these conditions, equations (31), (32) and (33) become

$$\left.\begin{aligned} H_\phi &= \frac{\epsilon \sin \Theta}{r} \frac{\ddot{f}(t - r/v)}{cv} \\ E_\Theta &= \frac{\sin \Theta}{r} \frac{\ddot{f}(t - r/v)}{v^2} \\ E_r &= 0 \text{ in comparison with } E_\Theta \end{aligned}\right\} \tag{42}$$

where $f(t)$ equals the moment of the doublet divided by the dielectric constant.

In *vacuum*, and sufficiently approximate in *air*,

$$\left.\begin{aligned} H_\phi &= \frac{\sin \Theta}{r} \frac{\ddot{f}(t - r/c)}{c^2} \\ E_\Theta &= \frac{\sin \Theta}{r} \frac{\ddot{f}(t - r/c)}{c^2} \\ E_r &= 0 \end{aligned}\right\} \tag{43}$$

where $f(t)$ = the moment of the doublet.

The electric and magnetic intensities, when the dielectric surrounding the oscillator is air, are equal to each other, and inversely proportional to the distance from the oscillator when this distance is large. The two intensities are directly proportional to the sine of the angle between the direction of the oscillator and the direction of the radius to the point under consideration. The electric intensity is in the direction of the meridional lines from the pole to the equator. The magnetic intensity is in the direction of the parallels of latitude.

76. Power Radiated through a Large Sphere.—If we consider a large sphere with the oscillator as center, we can apply Poynting's Theorem and obtain the power radiated through any surface element of the sphere or through the whole sphere.

The energy radiated per second (that is, the *power* radiated)

through an element of surface dS of the sphere is by (16), Chapter III,

$$\mathbf{u}_r dS = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}\, dS \tag{44}$$

$$= \frac{c}{4\pi} E_\Theta H_\phi\, dS \text{ numerically} \tag{45}$$

Substituting for E_Θ and H_ϕ their values from (43), we obtain

$$\mathbf{u}_r dS = \frac{\epsilon \sin^2\Theta}{4\pi r^2 v^3}\left\{\ddot{f}(t - r/v)\right\}^2 dS \tag{46}$$

with direction of r.

The element of surface

$$dS = r^2 \sin \Theta d\Theta d\phi.$$

This value substituted in (46) gives for the total power radiated through the sphere at great distance from the origin the value

$$\text{Total power radiated} = \frac{\epsilon \ddot{f}}{4\pi v^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin^3 \Theta d\Theta$$

$$= \frac{2}{3}\frac{\epsilon}{v^3}\left(\ddot{f}\right)^2 \tag{47}$$

where

$$f(t) = \frac{\text{moment of doublet}}{\epsilon} \tag{48}$$

and the $\ddot{f}$ in (47) is $\ddot{f}(t - r/v)$.

Equation (47) gives the total power (energy per second) passing through any distant sphere with the oscillator as center, and with an infinite medium of dielectric constant ϵ.

When the dielectric is *air*, (47) and (48) become

$$\text{Total power radiated} = \frac{2}{3c^3}\left(\ddot{f}\right)^2 \tag{49}$$

where

$$\ddot{f} = \ddot{f}\,(t - r/c),$$

and

$$f(t) = \text{moment of the doublet.}$$

77. Power Radiated by a Sinusoidal Oscillator in Air or Vacuum.—Let us next take the special case, in which the medium has unity dielectric constant and where the f of the dynamic electromagnetic field is assumed sinusoidal in the form

$$f = A \sin \omega(t - r/c).$$

In this case through a distant sphere by (49)

$$\text{Total power radiated} = \frac{2A^2\omega^4 \sin^2 \omega\,(t - r/c)}{3c^3},$$

of which the time average is

$$\bar{P} = \frac{A^2\omega^4}{3c^3}$$

$$= \frac{16\pi^4 A^2 c}{3\lambda^4} \tag{50}$$

where

$$\lambda = \text{wavelength} = \frac{2\pi c}{\omega}.$$

78. Radiation Resistance of Sinusoidal Oscillator.—For the oscillator described in the preceding section the moment of the oscillator is

$$f = A \sin \omega t = lq$$

where l is the length of the oscillator regarded as of the alternative type of Art. **71.** The current in such an oscillator is

$$i = \dot{q} = \frac{A\omega \cos \omega t}{l}$$

$$= \frac{2\pi c A \cos \omega t}{\lambda l}.$$

The mean square current is

$$\overline{i^2} = \frac{2\pi^2 c^2 A^2}{\lambda^2 l^2} \tag{51}$$

If we define the radiation resistance R of the oscillator as the mean power radiated divided by the mean square current, we have

$$R = \frac{8}{3}\frac{\pi^2 l^2}{c\lambda^2} \quad \text{E. S. units} \tag{52}$$

One electrostatic unit of resistance equals 9×10^{11} ohms, so that the radiation resistance in ohms becomes

$$R = \frac{80\pi^2 l^2}{\lambda^2} \text{ ohms} \tag{53}$$

Equation (53) *gives the radiation resistance of an oscillating doublet whose length l* (*or, as we have previously called it, p*) *is negligible in comparison with the wavelength λ of the radiated wave.*

The application of this formula to a radiotelegraphic antenna, as has been made by Rüdenberg,[1] is without theoretical justification, except in a very special case.

We shall, in the next chapter, discuss at length the radiation from a radiotelegraphic antenna.

[1] Rüdenberg: "Annalen der Physik," 25, p. 453.

CHAPTER IX

THEORETICAL INVESTIGATION OF THE RADIATION CHARACTERISTICS OF AN ANTENNA[1]

79. Introduction.—For the proper design of a radiotelegraphic transmitting station it is important to know the radiation characteristics of different types of antenna.

For example, if a flat-top antenna is to be employed, the question arises as to what is the best relation of the length of the horizontal part to the length of the vertical part, when the excitation is to be produced by a given type of generator. It may be known in a general way that the greater the vertical length, the greater the radiation resistance; it may also be known that the greater the horizontal length of the flat-top the greater the capacity of the antenna will be, and the greater will be the amount of current that can be made to flow from certain types of generator. Now these two quantities, radiation resistance and applied current, are both factors in determining the output from the antenna.

For a given generator, with known characteristics, the problem of getting the greatest output of high-frequency energy is a problem in the determination of the maximum value of the product of current square and radiation resistance of the antenna.

But this is not the whole problem, for there comes also into consideration the question as to how much of the radiated energy is radiated by the horizontal flat-top in what may be a useless direction.

Again, of the energy radiated from the vertical part of the antenna, how much of it contributes to the electric and magnetic forces on the horizon, where the receiving station is situated?

For the solution of these various problems it is important to know the radiation characteristics of the antenna in the form of certain functional relations. These relations should be known

[1] This chapter was originally published by the author in the Proceedings of the American Academy of Arts and Sciences, Vol. 52, pp. 192–252, 1916. Certain errors in the original publication are here corrected.

even when inductance is added at the base of the antenna for providing coupling or for increasing the wavelength to adapt it to the generator. These quantities should be known theoretically, since the ordinary measurements of these quantities do not permit us to distinguish radiation that is useful from the useless radiation as heat losses and from the radiation in useless directions.

It is the purpose of this chapter to give a treatment of this problem. Such a treatment is, so far as I know, up to the present entirely lacking, but the method here employed is that developed by Abraham[1] in a very remarkable paper entitled *Funkentelegraphie und Elektrodynamik.* In that paper, Abraham obtained theoretically the characteristics of a straight oscillator vibrating with its natural fundamental and harmonic frequencies. The present work is an extension of Abraham's method to the much more difficult problem of an antenna with a flat-top and with added inductance at the base.

80. Inadequacy of the Conception of an Antenna as a Doublet.—Apart from the brilliant investigation by Abraham, all other attempts at the treatment of the radiation from an antenna assume that the antenna is a Hertzian Doublet.[2] This is only a very crude approximation to the facts, for *the derivation of the electromagnetic field about a doublet assumes that the length of the doublet is negligible in comparison with a quantity that is itself negligible in comparison with the wavelength.*

Hence, the doublet theory will apply in all of its essentials to an antenna, only provided the length of the antenna is not greater than one three thousandth of the wavelength emitted (see Art. **74**). Of course, it may be that at great distances from the oscillator, the theory that it is a doublet may not introduce any large errors into certain problems such as the propagation over the surface of the earth; but the present treatment shows that the doublet theory does introduce large errors into computations of such quantities as the electric and magnetic field intensities and the radiation resistance of an antenna. It seems probable that other problems also should be revised in such a way as to replace the conception of the antenna as a doublet by the view of it as an oscillator that has a length comparable with one quarter of the wavelength.

81. Method of the Present Investigation.—In the present investigation, a doublet of infinitesimal length is assumed *at each*

[1] M Abraham. Physikalische Zeitschrift, **2,** 329–334 (1901).
[2] See Chapter VIII of present volume.

point of the antenna. This is the device used by Abraham These elementary doublets are free from the objection regarding their lengths, as they are of infinitesimal lengths, while the wavelength is that due to the whole antenna and therefore is enormously large in comparison with the lengths of the elemental doublets. The electric and magnetic forces due to each of the doublets is determined at a distant point and is summed up for all of the doublets of the antenna, *with strict regard to the difference of phase due to the different locations of the different doublets.* Such a process performed for all points of a distant sphere surrounding the antenna gives the total electric and magnetic forces at all points on the sphere. Then by integrating Poynting's Vector over the entire sphere, we obtain the total power radiated, and from this we compute the radiation resistance and other characteristics of the antenna.

The effect due to the vertical portion of the antenna and to the horizontal flat-top portion are computed separately, so as to give information as to how much energy is radiated with its electric force perpendicular to the horizon and how much parallel to the horizon.

In deciding as to the proper distribution of the elemental doublets along the antenna, the form of the current curve from point to point of the antenna is assumed independently. This process is not entirely above reproach, because Maxwell's equations, if they could be properly applied to the problem, would themselves give the distribution that is consistent with the applied electromotive force at the base of the antenna and with the shape and form of the antenna. This step of accurately deriving the distribution is, however, at the present time not possible of mathematical execution.

The distribution here assumed for the current in the antenna, as a function of the time and of the position along the antenna, and is given in the next section.

82. Assumed Current Distribution.—The form of antenna to which the whole discussion is devoted is illustrated in Fig. 1, and consists of a vertical portion of length a and a horizontal flat-top portion of length b. These quantities a and b may have any relative values whatever.

At the base of the antenna is an arbitrary inductance L for varying the wavelength.

The current at any point P' of the antenna is assumed to be given by the equation

$$i = I \sin\frac{2\pi c}{\lambda} t \cdot \sin \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - l\right) \tag{1}$$

where

c = velocity of light,
λ_0 = natural wavelength of the antenna without inductance,
λ = the wavelength with the inductance,
i = the current at the point P',
l = length measured along the antenna from the inductance to the point P'.

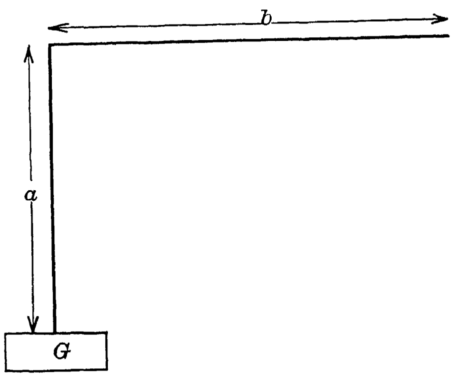

FIG. 1—Type of antenna. An inductance L not shown in this figure is supposed to be inserted between the antenna and the ground G for varying wavelength

The character of the assumed distribution is as follows: The factor $\sin \frac{2\pi c}{\lambda} t$ means that the current is sinusoidal in time at every point of the antenna, with the angular velocity

$$\omega = \frac{2\pi}{T} = \frac{2\pi c}{cT} = \frac{2\pi c}{\lambda} \tag{2}$$

The meaning of the other factor

$$I \sin \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - l\right) = J \text{ (say)} \tag{3}$$

is illustrated in the diagrams (a), (b) and (c) of Fig. 2.

If there is no inductance, $\lambda = \lambda_0$, and the factor becomes

$$J = I \cos \frac{2\pi l}{\lambda} \tag{4}$$

This is illustrated in (a).

In the case with added inductance, $\lambda \neq \lambda_0$, and we must keep the general form of J given in equation (3). This equation for positive values of l gives the upper half of the diagram (b). When l is supposed negative the curves obtained continue along the dotted lines of (b) and do not give a figure symmetrical with the upper half. *To produce proper symmetry the absolute value of l must be employed in equation* (1) *when it is applied to the distribution of the image to take account of reflection.*

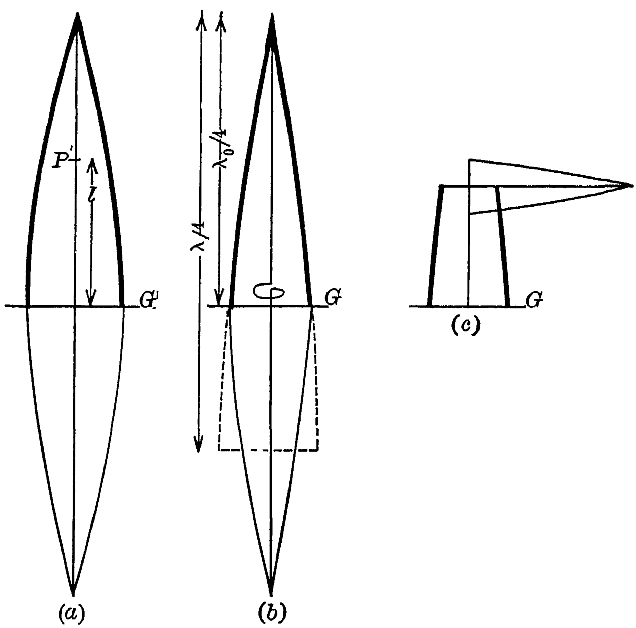

FIG. 2 —Assumed distribution of current in the antenna.

It is also to be carefully noted that when $l = 0$, equation (1) becomes

$$i_0 = I \sin \frac{\pi\lambda_0}{2\lambda} \sin \frac{2\pi c}{\lambda} t \tag{5}$$

so the amplitude at the base of the antenna is

$$I_0 = I \sin \frac{\pi\lambda_0}{2\lambda} \tag{6}$$

Now, finally, when the antenna has a flat-top it is assumed that the top part of the antenna is bent over without any significant

change in the magnitude of the current at the various points, as illustrated in (*c*).

When the equation (1) is to be applied to the vertical portion of the antenna, we shall call

$$l = z' \tag{7}$$

where

z' = vertical distance from the ground of the point P' on the antenna.

When the equation is to be applied to the horizontal part of the antenna, we shall call

$$l = a + x' \tag{8}$$

where

x' = distance along the horizontal part of the antenna to any point P'' on the flat-top.

The discussion will now be divided into several Parts: Part I. Electromagnetic Field Due to Vertical Portion of the Antenna; Part II. Field Due to Horizontal Portion of the Antenna; Part III. The Mutual Term in Power Determination. Part IV. Computations of Radiation Resistance. Part V. Field Intensities and Summary.

PART I

FIELD DUE TO VERTICAL PORTION OF ANTENNA

83. Coördinates.—Let the origin of coordinates be at the point of connection of the antenna to the ground. Let the z-axis be vertical. About this vertical axis as polar diameter, let us construct a system of spherical coordinates in which the position of any point P is given by its distance r_0 from the origin, and the angles θ and ϕ.

θ = the angle along meridional lines from the pole,
ϕ = the angle along parallels of latitude from a vertical plane of reference whose position is at present immaterial.

This system of coördinates with the positive directions of the angles indicated is given in Fig. 3.

If z' is the vertical ordinate of any point P' on the vertical portion of the antenna, and r the distance from P' to P, and if the distance OP is large in comparison with z', we may write (see Fig. 4)

$$r = r_0 - z' \cos \theta \tag{9}$$

84. Field Due to a Doublet at P′.—At a distant point P the electric and magnetic intensities due to a doublet of length dz' and charges e and $-e$ at P' are, by Hertz's theory, given in Art. **75**,

$$dE_\theta = dH_\varphi = \frac{\sin\theta}{c^2 r_0}\ddot{f}(t - r/c) \tag{10}$$

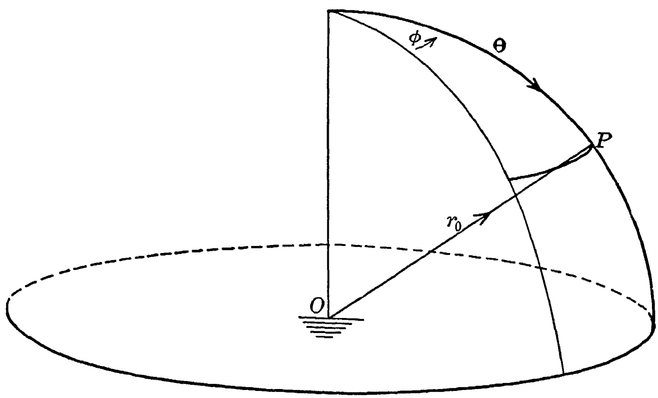

FIG. 3 —A set of spherical coordinates The coordinates of P are r_0, θ, ϕ.

where

$f(t)$ = the moment of the doublet
= $e\,dz'$, where e is in electrostatic units, (11)
dE_θ = the electric intensity in electrostatic units, which is entirely in the direction of θ; that is, of the meridional lines;
$dH\phi$ = the magnetic intensity in electromagnetic units, which is entirely in the direction of the parallels of latitude;
r = distance $P'P$ in centimeters,
c = velocity of light in centimeters per second.

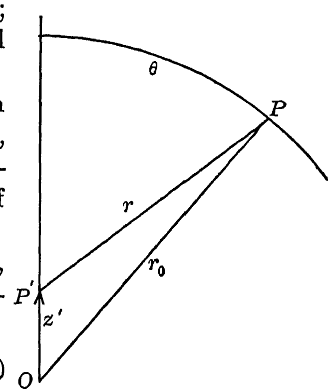

FIG. 4.

The two dots over the f in (10) indicate the second time derivative.

In writing equation (10), the slight difference in the direction of the perpendicular to r from the direction of the perpendicular to r_0 is neglected in view of the

largeness of r_0 in comparison with the length z' measured on the antenna.

Also the r which should occur in the denominator of (10) has been replaced by r_0, which can be done without appreciable error for large values of r. The same substitution cannot be made in the argument of f in (10), for there r determines the phase of the oscillation, and this phase changes through an angle of π for a half wavelength, independent of the distance from the origin.

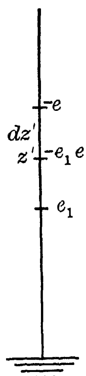

FIG. 5.

85. Expression of the Field in Terms of Current.—We shall next express the moment of the doublet and the intensities of the field in terms of the current i at the point z'. To do this we shall think of the current as delivering a charge $+ e$ to one end of the element of length dz' and a charge $- e$ to the other end of dz' in a certain time. A neighboring doublet has a different current and delivers different charges $+ e_1$ and $- e_1$ partly counteracting the charges of the given doublet, and leaving just the charge $e - e_1$ that actually occurs on the wire This is represented in Fig. 5.

With this view of the case, when i is in e.s.u.,

$$i = \dot{e},$$

and

$$\ddot{f}(t) = \dot{e}\, dz' = \frac{\partial i}{\partial t} dz' \tag{12}$$

Whence, by substituting the value of i from equation (1) into equation (12) we shall have, in view of (7) and (9)

$$dE_\theta = dH_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \cos\frac{2\pi}{\lambda}(ct - r_0 + z'\cos\theta)\cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz' \tag{13}$$

By integrating this expression from $z' = 0$ to $z' = a$, we obtain the electric and magnetic intensities at the point P due to direct transmission from the vertical portion of the antenna. Indicating this integration, we have

$$E_\theta = H_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0} \int_0^a \cos\frac{2\pi}{\lambda}(ct - r_0 + z'\cos\theta)\cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right) dz' \tag{14}$$

By reflection from the earth, which we shall regard as a perfect reflector, we have intensities that must be added to the above. These intensities may be obtained by considering the radiation to come from an image point at a distance z' below the surface. The effect of this is obtained by changing the sign of the z' in the cosine term of equation (14), but as was pointed out in Art. **82** the sign of z' in the sine term must remain. We obtain thus for the intensities due to the reflected wave emitted by the vertical portion of the antenna the value

$$E_\theta = H_\phi = \frac{2\pi I \sin\theta}{\lambda c r_0}\int_0^a \cos\frac{2\pi}{\lambda}(ct - r_0 - z'\cos\theta)\cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right)dz' \qquad (15)$$

Adding the equation (15) for the reflected intensities to the direct intensities of (14), remembering that if A and B are any two angles

$$\cos(A - B) + \cos(A + B) = 2\cos A \cos B \qquad (16)$$

we obtain for the total intensities at P the equation

$$E_\theta = H_\phi = \frac{4\pi I \sin\theta}{\lambda c r_0}\cos\frac{2\pi}{\lambda}(ct - r_0)\int_0^a \cos\left(\frac{2\pi z'}{\lambda}\cos\theta\right)\cdot \sin\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - z'\right)dz' \qquad (17)$$

which resolves into

$$E_\theta = H_\phi = \frac{4\pi I \sin\theta}{\lambda c r_0}\cos\frac{2\pi}{\lambda}(ct - r_0) \left[\sin\frac{\pi\lambda_0}{2\lambda}\int_0^a \cos\frac{2\pi z'\cos\theta}{\lambda}\cos\frac{2\pi z'}{\lambda}dz' - \cos\frac{\pi\lambda_0}{2\lambda}\int_0^a \cos\frac{2\pi z'\cos\theta}{\lambda}\sin\frac{2\pi z'}{\lambda}dz'\right] \qquad (18)$$

This expression may be integrated by the formulas 360 and 361 of B. O. Peirce's *Short Table of Integrals* and gives

$$E_\theta = H_\phi = \frac{2I}{cr_0\sin\theta}\cos\frac{2\pi}{\lambda}(ct - r_0) \{\cos B\cos(A\cos\theta) - \sin B\cos\theta\sin(A\cos\theta) - \cos G\} \qquad (19)$$

where

$$\left.\begin{aligned} B &= \frac{2\pi b}{\lambda} \\ A &= \frac{2\pi a}{\lambda} \\ G &= \frac{\pi\lambda_0}{2\lambda} = A + B \end{aligned}\right\} \qquad (20)$$

The quantity b, which is the length of the flat top, gets into (20) and (19) by reason of the fact that $a + b$ = the whole length of the antenna, so that

$$\lambda_0 = 4\ (a + b) \tag{21}$$

Equation (19), *with the notation of equations* (20) *and* (21) *is the general equation for the electric and magnetic intensities at any distant point P, due to the whole vertical part of the antenna.* In this formula, referring to Fig. 6,

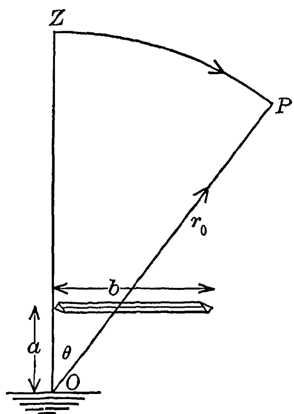

FIG. 6.

r_0 = the distance OP in cm.,
θ = the zenith angle ZOP,
b = length of the horizontal flat top in cm.,
a = length of vertical part of antenna, in cm.,
$\lambda_0 = 4\ (a + b)$ = natural wavelength in cm.,
λ = wavelength in cm. actually emitted, and differing from λ_0 by virtue of the added inductance,
I_0 = amplitude of current in absolute electrostatic units at the base of the antenna and related to I by the equation,

$$I_0 = I \sin \frac{\pi\lambda_0}{2\lambda}.$$

We shall reserve comment on this equation until after investigation of other characteristics of the radiation. See Part IV.

86. Total Power Radiated from the Vertical Part of the Antenna.—Having obtained in equation (19) the electric and magnetic intensities at any required point at a distance from the antenna, we shall next compute the total power radiated from the vertical part of the antenna, and shall then obtain its radiation resistance.

Since E_θ and H_ϕ are perpendicular to one another and perpendicular to r_0, we have, according to Poynting's theorem for the power radiated in the direction of r_0 through an element of surface dS perpendicular to r_0 the quantity

$$dp = \frac{c}{4\pi} E_\theta H_\phi dS \tag{22}$$

Let the element of surface be an elemental zone on the surface of the sphere, then

$$dS = 2\pi r_0^2 \sin \theta d\theta \tag{23}$$

This quantity, together with the values of E_θ and H_ϕ from (19), substituted in (22) and properly integrated, gives for the total power radiated through the whole hemisphere above the earth's surface, the value in ergs per second following:

$$\begin{aligned} p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[\cos^2 B\int_0^{\pi/2}\frac{\cos^2(A\cos\theta)\,d\theta}{\sin\theta} \\ + \sin^2 B\int_0^{\pi/2}\frac{\cos^2\theta\,\sin^2(A\cos\theta)\,d\theta}{\sin\theta} + \cos^2 G\int_0^{\pi/2}\frac{d\theta}{\sin\theta} \\ - 2\sin B\cos B\int_0^{\pi/2}\frac{\cos\theta\sin(A\cos\theta)\cos(A\cos\theta)\,d\theta}{\sin\theta} \\ - 2\cos B\cos G\int_0^{\pi/2}\frac{\cos(A\cos\theta)\,d\theta}{\sin\theta} \\ + 2\sin B\cos G\int_0^{\pi/2}\frac{\cos\theta\sin(A\cos\theta)\,d\theta}{\sin\theta}\Bigg] \end{aligned} \tag{24}$$

This equation when integrated gives the power radiated from the vertical part of the antenna. The integration is a tedious operation, and is given in the next section, which may be omitted by readers not interested in the mathematical processes involved. The result of the integration is found in Art. **88.**

87. The Integration of Equation (24).—By the use of such trigonometric equations as

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

the squares of sines and cosines in the integrands of (24) may be avoided, and equation (24) written

$$\begin{aligned} p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[\left(\frac{1}{2} + \cos^2 G\right)\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin\theta} \\ + \frac{\cos 2B}{2}\int_0^{\pi/2}\frac{\cos(2A\cos\theta)\,d\theta}{\sin\theta} - \frac{\sin^2 B}{2}\int_0^{\pi/2}\sin\theta\,d\theta \\ + \frac{\sin^2 B}{2}\int_0^{\pi/2}\sin\theta\cos(2A\cos\theta)\,d\theta \end{aligned}$$

$$- \frac{\sin 2B}{2}\int_0^{\pi/2} \frac{\cos\theta \sin(2A\cos\theta)\,d\theta}{\sin\theta}$$

$$- 2\cos B \cos G \int_0^{\pi/2} \frac{\cos(A\cos\theta)\,d\theta}{\sin\theta}$$

$$+ 2\sin B \cos G \int_0^{\pi/2} \frac{\cos\theta \sin(A\cos\theta)\,d\theta}{\sin\theta}\Bigg] \tag{25}$$

The third and fourth terms may be integrated directly. In the other terms let us introduce a change of variable as follows: Let

$$u = \cos\theta$$

$$d\theta = \frac{-du}{\sin\theta},$$

then

$$\int_0^{\pi/2} \frac{d\theta}{\sin\theta} = \int_1^0 \frac{-du}{1-u^2} = \frac{1}{2}\int_0^1 \left(\frac{1}{1+u} + \frac{1}{1-u}\right)du$$

$$= \frac{1}{2}\int_0^1 \frac{du}{1+u} + \frac{1}{2}\int_{-1}^0 \frac{du}{1+u} = \frac{1}{2}\int_{-1}^{+1} \frac{du}{1+u} \tag{26}$$

With this operation as a model, two of the other integrals of (25) may be written, respectively

$$\int_0^{\pi/2} \frac{\cos(2A\cos\theta)\,d\theta}{\sin\theta} = \frac{1}{2}\int_{-1}^{+1} \frac{\cos(2Au)\,du}{1+u} \tag{27}$$

$$\int_0^{\pi/2} \frac{\cos(A\cos\theta)\,d\theta}{\sin\theta} = \frac{1}{2}\int_{-1}^{+1} \frac{\cos(Au)\,du}{1+u} \tag{28}$$

Another of the integrals, examined in more detail, gives

$$\int_0^{\pi/2} \frac{\cos\theta \sin(2A\cos\theta)\,d\theta}{\sin\theta}$$

$$= \int_0^1 \frac{u\sin(2Au)du}{1-u^2}$$

$$= \frac{1}{2}\int_0^{+1}\left(\frac{1}{1-u} - \frac{1}{1+u}\right)\sin(2Au)\,du$$

$$= -\frac{1}{2}\int_0^1 \frac{\sin(2Au)\,du}{1+u} + \frac{1}{2}\int_0^{-1} \frac{\sin(2Au)\,du}{1+u}$$

$$= -\frac{1}{2}\int_{-1}^{+1} \frac{\sin(2Au)\,du}{1+u} \tag{29}$$

Similarly, the remaining integral becomes

$$\int_0^{\pi/2} \frac{\cos\theta \sin(A\cos\theta)\,d\theta}{\sin\theta} = -\frac{1}{2}\int_{-1}^{+1}\frac{\sin(Au)\,du}{1+u} \tag{30}$$

Returning now to equation (25), we shall integrate the third and fourth terms, setting them first, and shall substitute (26) to (30) for the other terms, obtaining

$$p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[-\frac{\sin^2 B}{2} + \frac{\sin^2 B}{2}\frac{\sin 2A}{2A}$$
$$+\left(\frac{1}{2} + \cos^2 G\right)\frac{1}{2}\int_{-1}^{+1}\frac{du}{1+u}$$
$$+\frac{\cos 2B}{4}\int_{-1}^{+1}\frac{\cos(2Au)\,du}{1+u} + \frac{\sin 2B}{4}\int_{-1}^{+1}\frac{\sin(2Au)\,du}{1+u}$$
$$-\cos G\left\{\cos B\int_{-1}^{+1}\frac{\cos(Au)\,du}{1+u} + \sin B\int_{-1}^{+1}\frac{\sin(Au)\,du}{1+u}\right\}\Bigg] \tag{31}$$

Let us now write

$$\gamma = 2A(1+u), \tag{32}$$
$$2Au = \gamma - 2A,$$
$$du = \frac{d\gamma}{2A},$$
$$\frac{du}{1+u} = \frac{d\gamma}{\gamma};$$

then the second and third integrals of (31) become

$$\frac{\cos 2B}{4}\int_{-1}^{+1}\frac{\cos(2Au)\,du}{1+u} + \frac{\sin 2B}{4}\int_{-1}^{+1}\frac{\sin(2Au)\,du}{1+u}$$
$$= \frac{\cos 2B}{4}\int_0^{4A}\{\cos\gamma\cos 2A + \sin\gamma\sin 2A\}\frac{d\gamma}{\gamma}$$
$$+\frac{\sin 2B}{4}\int_0^{4A}\{\sin\gamma\cos 2A - \cos\gamma\sin 2A\}\frac{d\gamma}{\gamma}$$
$$= \frac{\cos(2A+2B)}{4}\int_0^{4A}\frac{\cos\gamma\,d\gamma}{\gamma} + \frac{\sin(2A+2B)}{4}\int_0^{4A}\frac{\sin\gamma\,d\gamma}{\gamma}$$
$$= \frac{\cos 2G}{4}\int_0^{4A}\frac{\cos\gamma}{\gamma}\,d\gamma + \frac{\sin 2G}{4}\int_0^{4A}\frac{\sin\gamma}{\gamma}\,d\gamma.$$

In like manner, the last line of (31) becomes

$$-\cos^2 G\int_0^{2A}\frac{\cos\gamma}{\gamma}\,d\gamma - \cos G\sin G\int_0^{2A}\frac{\sin\gamma}{\gamma}\,d\gamma \tag{33}$$

Let us now decompose the coefficient of the first integral of (31) as follows:

$$\frac{1}{4} + \frac{\cos^2 G}{2} = \frac{1}{4} + \cos^2 G - \frac{\cos^2 G}{2}$$

$$= \frac{1}{4} - \frac{1 + \cos 2G}{4} + \cos^2 G$$

$$= -\frac{\cos 2G}{4} + \cos^2 G.$$

Then the whole equation (31) may be written

$$p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[-\frac{\sin^2 B}{2} + \frac{\sin^2 B \sin 2A}{4A}$$
$$-\frac{\cos 2G}{4}\int_o^{4A}\frac{(1-\cos\gamma)d\gamma}{\gamma} + \frac{\sin 2G}{4}\int^{4A}\frac{\sin\gamma}{\gamma}\,d\gamma$$
$$+\cos^2 G\int_o^{2A}\frac{(1-\cos\gamma)d\gamma}{\gamma} - \frac{\sin\ 2G}{2}\int_o^{2A}\frac{\sin\gamma}{\gamma}\,d\gamma\Bigg] \qquad (34)$$

The various integrals may now be obtained by expanding in series and integrating term by term. This gives

$$p = \frac{2I^2}{c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[\frac{\sin^2 B}{2}\left(\frac{\sin 2A}{2A} - 1\right)$$
$$-\frac{\cos 2G}{4}\left\{\frac{(4A)^2}{2!2} - \frac{(4A)^4}{4!4} + \frac{(4A)^6}{6!6} - \cdots\right\}$$
$$+\frac{1+\cos 2G}{2}\left\{\frac{(2A)^2}{2!2} - \frac{(2A)^4}{4!4} + \frac{(2A)^6}{6!6} - \cdots\right\}$$
$$+\frac{\sin 2G}{4}\left\{4A - \frac{(4A)^3}{3!3} + \frac{(4A)^5}{5!5} - \cdots\right\}$$
$$-\frac{\sin 2G}{2}\left\{2A - \frac{(2A)^3}{3!3} + \frac{(2A)^5}{5!5} - \cdots\right\}\Bigg] \qquad (35)$$

Let us now eliminate B from the first terms of this equation, by substituting $B = G - A$, obtaining

$$\frac{\sin^2 B}{2}\left(\frac{\sin 2A}{2A} - 1\right) = \frac{1 - \cos 2B}{4}\left(\frac{\sin 2A}{2A} - 1\right)$$

$$= \left\{\frac{1}{4} - \frac{\cos(2G - 2A)}{4}\right\}\left(\frac{\sin 2A}{2A} - 1\right)$$

$$= -\frac{1}{4} + \frac{\cos 2G \cos 2A}{4} + \frac{\sin 2G \sin 2A}{4}$$

$$+ \frac{\sin 2A}{8A} - \frac{\cos 2G \sin 4A}{16A}$$

$$- \frac{\sin 2G}{4}\frac{1 - \cos 4A}{4A} \tag{36}$$

If now we expand in series the quantities involving A in (36) and substitute in (35), we obtain, if

$$\left.\begin{aligned} k &= 2A \\ q &= 2G \end{aligned}\right\} \tag{37}$$

$$\begin{aligned} p = \frac{2I^2}{c} \cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\Bigg[\frac{1}{4}\left\{-\frac{k^2}{3!} + \frac{k^4}{5!} - \frac{k^6}{7!} + \cdot\right\} \\ + \frac{1}{4}\left\{\frac{2k^2}{2!2} - \frac{2k^4}{4!4} + \frac{2k^6}{6!6} - \cdot\cdot\right\} \\ + \frac{\cos q}{4}\left\{1 - \frac{k^2}{2!} + \frac{k^4}{4!} - \frac{k^6}{6!} + \cdot\cdot\right\} \\ + \frac{\sin q}{4}\left\{k - \frac{k^3}{3!} + \frac{k^5}{5!} - \frac{k^7}{7!} + \cdots\right\} \\ - \frac{\cos q}{4}\left\{1 - \frac{(2k)^2}{3!} + \frac{(2k)^4}{5!} + \cdot\cdot\right\} \\ - \frac{\sin q}{4}\left\{\frac{2k}{2!} - \frac{(2k)^3}{4!} + \frac{(2k)^5}{6!} - \cdots\right\} \\ - \frac{\cos q}{4}\left\{\frac{(2k)^2}{2!2} - \frac{(2k)^4}{4!4} + \frac{(2k)^6}{6!6} - \cdots\right\} \\ + \frac{\cos q}{2}\left\{\frac{k^2}{2!2} - \frac{k^4}{4!4} + \frac{k^6}{6!6} - \cdots\right\} \\ + \frac{\sin q}{4}\left\{2k - \frac{(2k)^3}{3!3} + \frac{(2k)^5}{5!5} - \cdots\right\} \\ - \frac{\sin q}{2}\left\{k - \frac{k^3}{3!3} + \frac{k^5}{5!5} - \cdots\right\}\Bigg] \end{aligned} \tag{38}$$

If now we add together the terms multiplied by sin q and those multiplied by cos q, and those not involving q, we have (on factoring out the ¼)

$$p = \frac{I^2}{2c}\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\left[\left\{\frac{2+2}{3!2}k^2 - \frac{4+2}{5!4}k^4 + \frac{6+2}{7!6}k^6 - \cdots\right\}\right.$$
$$+ \cos q\left\{-\frac{2^2+2^2-4}{3!2}k^2 + \frac{4^2+2^4-6}{5!4}k^4 - \frac{6^2+2^6-8}{7!6}k^6 + \frac{8^2-2^8-10}{9!8}k^8 - \quad\right\}$$
$$\left. + \sin q\left\{-\frac{3^2+2^3-5}{4!3}k^3 + \frac{5^2+2^5-7}{6!5}k^5 - \frac{7^2+2^7-9}{8!7}k^7 + \frac{9^2+2^9-11}{10!9}k^9 - \quad \cdot\right\}\right] \tag{39}$$

Equation (39) *gives the total power radiated by the vertical portion of the antenna into the hemisphere above the earth's surface.* In this equation, the current factor I is in absolute c.g s. electrostatic units, and the power p is in ergs per second.

It is convenient to change the current factor into amperes and the radiated power into watts, which can be done by multiplying the right-hand side of (39) by 30 c. This is done, and the equation is rewritten in the next section.

88. Result of the Integration for Power Radiated from the Vertical Part of the Antenna.—By equation (39), when reduced to practical units, the total power radiated into the aërial hemisphere from the vertical part of the antenna may be written

$$p = I^2\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}\left[R_1 - R_2\cos q - R_3\sin q\right] \tag{40}$$

where

$$\left.\begin{aligned}
R_1 &= 15\left\{\frac{2+2}{3!2}k^2 - \frac{4+2}{5!4}k^4 + \frac{6+2}{7!6}k^6 - \cdots\right\}\\
R_2 &= 15\left\{\frac{2^2+2^2-4}{3!2}k^2 - \frac{4^2+2^4-6}{5!4}k^4 + \frac{6^2+2^6-8}{7!6}k^6 - \cdots\right\}\\
R_3 &= 15\left\{\frac{3^2+2^3-5}{4!3}k^3 - \frac{5^2+2^5-7}{6!5}k^5 + \frac{7^2+2^7-9}{8!7}k^7 - \cdots\right\}
\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} q &= \frac{\pi\lambda_0}{\lambda} \\ k &= \frac{4\pi a}{\lambda} \end{aligned}\right\} \qquad (42)$$

a = length of vertical part of antenna in same unit as λ (*e.g.*, meters),

p = radiated power in watts instantaneous value,

$$I = \frac{I_0}{\sin q/2} \qquad (43)$$

where

I_0 = amplitude of current at the base of antenna in amperes.

89. Radiation Resistance of Vertical Part of the Antenna. In equation (40) is given the power radiated from the vertical part of the antenna, on the assumption that radiation from the horizontal part of the antenna does not interfere with it. It will be shown later in §94 *et seq.* how this interference is computed and allowed for. Accepting for the present the assumption of non-interference, we may obtain the radiation resistance of the vertical part of the antenna.

The radiation resistance is defined as the *time average of radiated power divided by the time average of the square of the current at the base of the antenna.*

In taking the time average of the power (40), it is to be noted that the time average of $\cos^2\left\{\frac{2\pi}{\lambda}(ct - r_0)\right\}$ is ½. The time average of current square at the base of the antenna, by (1) is $\frac{1}{2}I^2\sin^2\frac{\pi\lambda_0}{2\lambda} = \frac{1}{2}I^2\sin^2\left(\frac{q}{2}\right)$. Whence the radiation resistance becomes in ohms

$$R_\Omega = \frac{1}{\sin^2\left(\frac{q}{2}\right)}\left\{R_1 - R_2\cos q - R_3\sin q\right\} \qquad (44)$$

in which R_1, R_2, R_3 and q have the values in (41) and (42).

We shall later give tables of R_1, R_2, and R_3, that will reduce the calculations of R to very simple operations, and shall compare the results with calculations on the doublet hypothesis and with observations.

We shall, however, first investigate theoretically the radiation from the horizontal part of the antenna. This is a problem of considerable mathematical difficulty but is capable of solution.

PART II

FIELD DUE TO HORIZONTAL PORTION OF ANTENNA

90. Introductory Notions.—To determine the electromagnetic field and radiation characteristics of the horizontal flat-top portion of the antenna, let the rectangular coördinates of any distant point P (Fig. 7) be x, y, z.

And let the coordinates of any point P' on the flat-top be x', 0, a; the coördinates of the image point P'' be x', 0, $-a$.

Then the distance from the origin of coordinates to the distant point is

$$OP = r_0 = \sqrt{x^2 + y^2 + z^2}$$

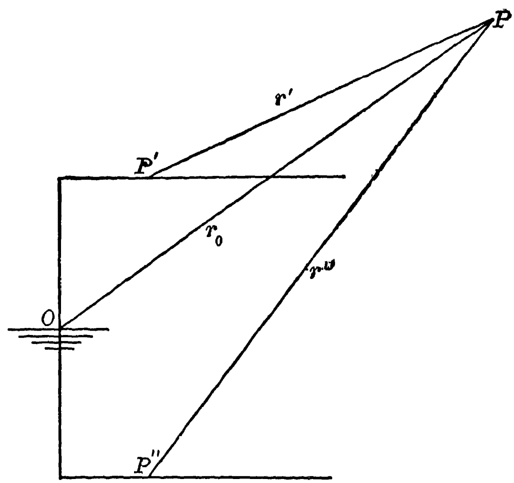

FIG. 7.

The distances of the distant point from the point on the flat-top and its image respectively are

$$P'P = r' = \sqrt{(x - x')^2 + y^2 + (z - a)^2}$$

and

$$P''P = r'' = \sqrt{(x - x')^2 + y^2 + (z + a)^2}.$$

Then

$$r' - r_0 = \sqrt{(x - x')^2 + y^2 + (z - a)^2} - \sqrt{x^2 + y^2 + z^2}.$$

As an approximation, let us multiply by the sum of these radicals and divide by the approximate value of this sum for large values of r_0; namely, by $2r_0$, obtaining

$$r' = r_0 + \frac{x'^2 - 2xx' - 2za + a^2}{2r_0} \tag{45}$$

$$r'' = r_0 + \frac{x'^2 - 2xx' + 2za + a^2}{2r_0} \tag{46}$$

91. Determination of Electric and Magnetic Intensities due to Flat-top.—The values of r' and r'' in (45) and (46) may be replaced by r_0 in intensity factors, but not in phase terms, and give for the sum of the effects of a doublet at P' and another at P'' (the image doublet) the electric and magnetic intensities

$$dE_\psi = dH_\Sigma = \frac{\sin\psi}{r_0c^2}\left\{f_1(t - r'/c) + f_2(t - r''/c)\right\} \tag{47}$$

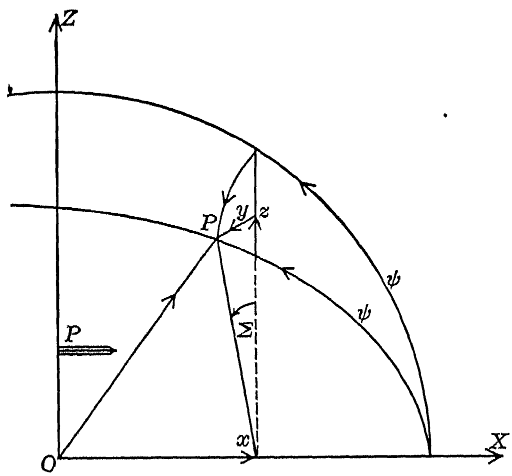

FIG. 8.

where $f_1(t)$ and $f_2(t)$ are the moments of the two doublets respectively. The angles φ and Σ correspond to the angles θ and ϕ of Fig. 3, except that the figure is turned on its side, so as to put the polar diameter along the x-axis instead of the z-axis. This arrangement is shown in Fig. 8. The plane of the zero value of Σ is now to be fixed as the plane of the x and z-axes.

Now using the current distribution of equation (1), we must replace l by $a + x'$, which gives, when treated as (12) was treated,

$$\ddot{f}_1 = \frac{2\pi cI}{\lambda}\cos\left\{\frac{2\pi}{\lambda}\left((ct - r_0 - \frac{x'^2 - 2x'x - 2za + a^2}{2r_0}\right)\right\}$$

$$\sin\left\{\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right)\right\}dx' \tag{48}$$

The fictitive current at P'' is just equal and opposite to that at P', with, however, a different distance from the point P, so we may write

$$f_2 = -\frac{2c\pi I}{\lambda}\cos\left\{\frac{2\pi}{\lambda}\left(ct - r_0 - \frac{x'^2 - 2xx' + 2za + a^2}{2r_0}\right)\right\}$$
$$\sin\left\{\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right)\right\}dx'. \tag{49}$$

Whence by addition, employing the trigonometric relation

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta,$$

equation (47) becomes

$$dE_\psi = dH_\Sigma = -\frac{4\pi I\sin\psi}{r_0 c\lambda}\sin\left\{\frac{2\pi}{\lambda}\left(ct - r_0 - \frac{x'^2 - 2xx' + a^2}{2r_0}\right)\right\}$$
$$\left[\sin\frac{2\pi az}{\lambda r_0}\sin\left\{\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right)\right\}dx'\right].$$

In this equation we may as usual replace $\frac{2\pi a}{\lambda}$ by A. Also we may make an approximation as follows: For large values of r_0

$$\frac{x'^2 - 2xx' + a^2}{2r_0} = -\frac{xx'}{r_0} = -x'\cos\psi.$$

In making this approximation the neglected term is $\frac{x'^2 + a^2}{2r_0}$, and this is to be neglected even in the phase angle, because its value is absolutely small. We have then

$$dE_\psi = dH_\Sigma = -\frac{4\pi I\sin\psi}{r_0 c\lambda}\sin\frac{Az}{r_0}\sin\left\{\frac{2\pi}{\lambda}(ct - r_0 + x'\cos\psi)\right\}$$
$$\left[\sin\left\{\frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a - x'\right)\right\}dx'\right] \tag{50}$$

This equation may be shortened up by writing

$$\tau = \frac{2\pi}{\lambda}(ct - r_0) \tag{51}$$

and

$$B = \frac{2\pi}{\lambda}\left(\frac{\lambda_0}{4} - a\right) = \frac{2\pi b}{\lambda} \tag{52}$$

To obtain the total electric and magnetic intensities due to the flat-top, the equation (50) must be integrated for all the doublets and their images between the limits

$$x' = 0 \text{ and } x' = b$$

where b is the length of the flat-top. This integration is expressed in the following equation.

$$E_\psi = H_\Sigma = -\frac{4\pi I \sin\psi}{r_0 c\lambda}\sin\frac{Az}{r_0}\int_o^b \sin\left(\tau + \frac{2\pi x'}{\lambda}\cos\psi\right)\sin\left(B - \frac{2\pi x'}{\lambda}\right)dx' \quad (53)$$

To perform the integration let us introduce a change of variable by putting

$$s = B - \frac{2\pi x'}{\lambda} \qquad \text{then } dx' = -\frac{\lambda}{2\pi}ds$$

and the limits of integration become

$$\text{for } x' = 0, \quad s = B, \qquad \text{for } x' = b, \quad s = 0.$$

Equation (53) then becomes

$$E_\psi = H_\Sigma = \frac{2I\sin\psi}{r_0 c}\sin\frac{Az}{r_0}\int_B^o \sin(\tau + B\cos\psi - s\cos\psi)\sin s\,ds$$
$$= \frac{2I\sin\psi}{r_0 c}\sin\frac{Az}{r_0}\left[\sin(\tau + B\cos\psi)\int_B^o \cos(s\cos\psi)\sin s\,ds - \cos(\tau + B\cos\psi)\int_B^o \sin(s\cos\psi)\sin s\,ds\right] \quad (54)$$

The expressions of this equation may be integrated by the use of formulas 360 and 359 of B. O. Peirce's Tables and give

$$E_\psi = H_\Sigma = \frac{2I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\left\{\sin(\tau + B\cos\psi)\Big[-\cos s\cos(s\cos\psi) - \cos\psi\sin s\sin(s\cos\psi)\Big]_B^0 - \cos(\tau + B\cos\psi)\Big[\cos\psi\sin s\cos(s\cos\psi) - \cos s\sin(s\cos\psi)\Big]_B^0\right.$$
$$= \frac{2I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\left[\sin(\tau + B\cos\psi)\left\{-1 + \cos B\cos\Big(B(\cos\psi)\Big) + \cos\psi\sin B\sin(B\cos\psi)\right\} + \cos(\tau + \mathrm{B}\cos\psi)\left\{\cos\psi\sin B\cos(B\cos\psi) - \cos B\sin(B\cos\psi)\right\}\right]$$
$$= \frac{2I}{r_0 c\sin\psi}\sin\frac{Az}{r_0}\left[\sin\tau\left\{\cos B - \cos(B\cos\psi)\right\} + \cos\tau\left\{\cos\psi\sin B - \sin(B\cos\psi)\right\}\right] \quad (55)$$

Equation (55) *gives the electric and magnetic intensities due to the flat-top at any distant point whose coordinates are*

r_0 = distance of the point from the origin,
z = vertical height of the point above the earth's surface,
ψ = angle between r_0 and the x-axis; this x-axis being parallel to the flat-top.

The quantities, A, B, and τ are defined by equations (20) and (51). We shall next discuss the total power radiated from the antenna.

92. Concerning Power Radiated from the Total Antenna.—It is to be noticed that the electric and magnetic intensities due to the flat-top of the antenna and those intensities due to the vertical portions of the antenna are directed along the meridional and latitudinal lines of two systems of polar coordinates with their poles one quadrant apart. This does not make the respective intensities perpendicular to each other, and it becomes necessary to resolve one set of these intensities along and perpendicular to the other set of intensities. At a given point on the sphere about the origin of coordinates, the quantities ϕ, θ, Σ and ψ are oriented in a manner represented in Fig. 9.

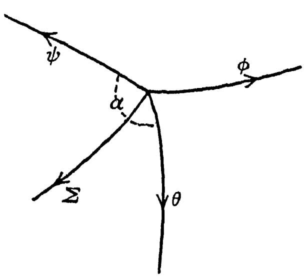

FIG. 9

If we let

$$\alpha = \text{angle between } \psi \text{ and } \theta$$

then also

$$\alpha = \text{angle between } \phi \text{ and } \Sigma.$$

It is also apparent that

$$\text{Angle between } \Sigma \text{ and } \theta = \alpha - \frac{\pi}{2}$$

$$\text{Angle between } \psi \text{ and } \phi = \frac{3\pi}{2} - \alpha$$

Let us now resolve E_ψ and H_Σ into components along θ and perpendicular thereto (that is, along ϕ) obtaining for the θ-components

$$E_{\psi,\theta} = E_\psi \cos \alpha$$
$$H_{\Sigma,\theta} = H_\Sigma \cos\left(\alpha - \frac{\pi}{2}\right) = H_\Sigma \sin \alpha,$$

and for the ϕ-components

$$E_{\psi,\varphi} = E_\psi \cos\left(\frac{3\pi}{2} - \alpha\right) = - E_\psi \sin\alpha$$

$$H_{\Sigma,\varphi} = H_\Sigma \cos\alpha.$$

Adding these quantities to the corresponding components of the intensities due to the vertical part of the antenna, we obtain for the total intensities, which are designated by primes, the values

$$\begin{aligned} E'_\theta &= E_\theta + E_\psi \cos\alpha, \\ E'_\varphi &= -E_\psi \sin\alpha, \\ H'_\theta &= H_\Sigma \sin\alpha \\ H'_\varphi &= H_\varphi + H_\Sigma \cos\alpha. \end{aligned}$$

All of these intensities are perpendicular to r_0. To get the power radiated through an element of surface dS perpendicular to r_0, we may make use of Poynting's vector, in the form

$$dp = \frac{c}{4\pi}\left(\mathbf{E}' \times \mathbf{H}'\right) dS$$

where the *cross* between the vectors means the vector-product. This vector-product, expanded, gives

$$\begin{aligned} dp &= \frac{c}{4\pi}\left(E'_\theta H'_\varphi - E'_\varphi H'_\theta\right) dS \\ &= \frac{c}{4\pi}\Big(E_\theta H_\varphi + E_\psi H_\Sigma \cos^2\alpha + H_\varphi E_\psi \cos\alpha + E_\theta H_\Sigma \cos\alpha + \\ &\qquad\qquad E_\psi H_\Sigma \sin^2\alpha\Big) dS \\ &= \frac{c}{4\pi}\left(E_\theta H_\varphi + E_\psi H_\Sigma + 2\cos\alpha\, E_\theta H_\Sigma\right) dS \qquad (56) \end{aligned}$$

We have already found the first term of this power and have obtained its integral all over the aërial hemisphere. This integral we have called *the power radiated from the vertical part* of the antenna. We shall call the second term above (56), when properly integrated, *the power radiated from the flat-top.* The third term, since it contains both sets of coördinates, may be called *power radiated mutually* These designations are merely for convenience in paragraphing the mathematics involved.

93. Power Radiated from the Flat-top.—Let us now enter upon a determination of the power contributed by the second term of the right-hand side of equation (56), and integrate this

term over the aërial hemisphere; that is, the hemisphere above the surface of the earth regarded as a plane.

The element of area of this hemisphere is

$$dS = r_0{}^2 \sin\psi\, d\psi\, d\Sigma \tag{57}$$

This is to be substituted in the required term involving E_Ψ and H_Σ; but these quantities involve the coördinate z, which must be replaced by its value in polar coördinates

$$z = r_0 \sin\psi \cos\Sigma \tag{58}$$

Besides (57) and (58) we are also to substitute the values of E_Ψ and H_Σ from (55) into the term

$$dp = \frac{c}{4\pi}\left(E_\psi H_\Sigma\right) dS \tag{59}$$

E_ψ and H_Σ are identical, by (55); the product will give certain terms involving $\sin^2\tau$, other terms involving $\cos^2\tau$, and still other terms involving $\sin\tau\cos\tau$; where τ has the value given in (51). If we take the time average for a complete cycle, or, if we prefer, for a time that is large in comparison with a complete period, we have

$$\text{av. } \sin^2\tau = \text{av. } \cos^2\tau = \tfrac{1}{2};$$

while the average of the product

$$\text{av. } \sin\tau\cos\tau = 0.$$

The integral form of (59) then becomes, if $\bar{p}$ = the time average of radiated power,

$$\bar{p} = \frac{I^2}{2\pi c}\int_0^\pi \frac{d\psi}{\sin\psi}\left[\left\{\cos^2 B + \cos^2\psi \sin^2 B + 1 - 2\cos B \cos(B\cos\psi) - 2\cos\psi \sin B \sin(B\cos\psi)\right\}\int_{-\pi/2}^{+\pi/2} d\Sigma[\sin^2(A\sin\psi\cos\Sigma)]\right] \tag{60}$$

We shall first perform the integration with respect to Σ

$$\int_{-\pi/2}^{\pi/2} d\Sigma\{\sin^2(A\sin\psi\cos\psi)\} = \int_{-\pi/2}^{\pi/2} d\Sigma\left\{\frac{1-\cos(2A\sin\psi\cos\Sigma)}{2}\right\}$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(2A\sin\psi\cos\Sigma)d\Sigma$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_{-\pi/2}^{0}\cos(2A\sin\psi\cos\Sigma)\,d\Sigma - \frac{1}{2}\int_0^{\pi/2}\cos(2A\sin\psi\cos\Sigma)d\Sigma \tag{61}$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi}\cos(2A\sin\psi\cos\Sigma)d\Sigma \tag{62}$$

This last step consists in changing the variable of the first integral of the right-hand side of (61) by putting

$$\Sigma' = \pi + \Sigma,$$

which makes the limits $\frac{\pi}{2}$ and π without any other change, except the change of Σ to Σ'. But since this is the variable of integration, the prime may be omitted, and the terms of (61) added, giving (62).

Equation (62) may now be integrated for Formula (11), Art. 121 of Byerly's *Fourier's Series and Spherical Harmonics* giving for the integral of (62)

$$\int_{-\pi/2}^{\pi/2} d\Sigma \{\sin^2 (A \sin \psi \cos \Sigma)\} = \frac{\pi}{2} - \frac{\pi}{2} J_0(2A \sin \psi) \qquad (63)$$

where J_0 is the Bessel's Function of the zeroth order, with a development of the form

$$J_0 (x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \ . \ . \ . \qquad (64)$$

Before substituting in (60) let us simplify the general trigonometric factor in the brace of (60) by placing $\cos^2 \psi = 1 - \sin^2 \psi$, and letting $k = 2A$, as in (42), we then obtain

$$\bar{p} = \frac{I^2}{4c}\int_0^\pi \left\{\frac{1 - J_0 (k \sin \psi)}{\sin \psi}\right\} \left\{2 - \sin^2 \psi \sin^2 B \right.$$
$$\left. - 2 \cos B \cos (B \cos \psi) - 2 \cos \psi \sin B \sin (B \cos \psi) \right\} d\psi$$
$$= \frac{I^2}{4c}\int_0^\pi \left\{\frac{k^2 \sin \psi}{2^2} - \frac{k^4 \sin^3 \psi}{2^2 4^2} + \frac{k^6 \sin^5 \psi}{2^2 4^2 6^2} - \cdots \right\}$$
$$\left\{2 - \sin^2 \psi \sin^2 B - 2 \cos B \cos (B \cos \psi) \right.$$
$$\left. - 2 \cos \psi \sin B \sin (B \cos \psi) \right\} d\psi, \qquad (65)$$

or

$$\bar{p} = \frac{I^2}{4c}\Big[- 2 \sum (- 1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \ldots n^2}\int_0^\pi \sin^{n-1} \psi d\psi$$
$$+ \sin^2 B \sum (- 1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \ldots n^2}\int_0^\pi \sin^{n+1} \psi d\psi$$
$$+ 2 \cos B \sum (- 1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \ldots n^2}\int_0^\pi \sin^{n-1} \psi \cos (B \cos \psi) d\psi$$

$$+ 2 \sin \sum B(-1)^{\frac{n}{2}} \frac{k^n}{2^2 4^2 6^2 \quad n^2} \int_o^\pi \sin^{n-1} \psi \cos \psi \sin (B \cos \psi) \, d\psi \Big]$$

$$n = 2, 4, 6, \ldots \tag{66}$$

Treating these several integrals separately, we have

$$\int_o^\pi \sin^{n-1} \psi d\psi = \int_o^{\frac{\pi}{2}} \sin^{n-1} \psi d\psi + \int_{\frac{\pi}{2}}^\pi \sin^{n-1} \psi d\psi$$

$$= \int_o^{\frac{\pi}{2}} \sin^{n-1} \psi d\psi + \int_o^{\frac{\pi}{2}} \cos^{n-1} \psi d\psi$$

$$= 2\left\{\frac{2 \cdot 4 \cdot 6 \cdots n - 2}{1 \cdot 3 \cdot 5 \cdots n - 1}\right\} \tag{67}$$

by B. O. Peirce's Tables, Formula No. 483.

Likewise

$$\int_o^\pi \sin^{n+1} \psi d\psi = 2\left\{\frac{2 \cdot 4 \cdot 6 \cdots n}{1 \cdot 3 \cdot 5 \cdots n + 1}\right\} \tag{68}$$

Now by Byerly's *Fourier's Series and Spherical Harmonics* equation (9), Art. 121,

$$\int_o^\pi \sin^{n-1} \psi \cos (B \cos \psi) \, d\psi = \frac{2^{\frac{n-1}{2}} \sqrt{\pi} \Gamma\left(\frac{n}{2}\right)}{B^{\frac{n-1}{2}}} J_{\frac{n-1}{2}} (B) \tag{69}$$

where $J_{\frac{n-1}{2}} (B)$ is a Bessel's Function of the order $(n - 1)/2$, and $\Gamma\left(\frac{n}{2}\right)$ is the Gamma Function of $\frac{n}{2}$.

For the last integral of (66), we have by Problem 2 and equation (9) of the same article of Byerly's *Fourier's Series*

$$\int_o^\pi \sin^{n-1} \psi \cos \psi \sin (B \cos \psi) \, d\psi$$

$$= \frac{B}{n} \int_o^\pi \sin^{n+1} \psi \cos (B \cos \psi) \, d\psi$$

$$= \frac{B}{n} \cdot \frac{2^{\frac{n+1}{2}} \sqrt{\pi} \Gamma\left(\frac{n}{2} + 1\right)}{B^{\frac{n+1}{2}}} J_{\frac{n+1}{2}} (B) \tag{70}$$

Substituting these various integrations (67), (68), (69), and (70) in (66), we have

$$\bar{p} = \frac{1^2}{4c}\Big[\sum -4(-1)^{\frac{n}{2}}\frac{k^n}{n!n} + 2\sin^2 B\sum(-1)^{\frac{n}{2}}\frac{k^n}{n+1!}$$

$$+ 2\cos B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2\,4^2\cdots n^2}\;\frac{2^{\frac{n-1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{B^{\frac{n-1}{2}}}J_{\frac{n-1}{2}}(B)$$

$$+ 2\sin B\sum(-1)^{\frac{n}{2}}\frac{k^n}{2^2\,4^2\cdots n^2}\frac{B}{n}\;\frac{2^{\frac{n+1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)}{B^{\frac{n+1}{2}}}J_{\frac{n+1}{2}}(B) \tag{71}$$

$n = 2, 4, 6, \cdots\infty$ $\qquad B = \frac{2\pi b}{\lambda}$ is between 0 and $\frac{\pi}{2}$.

This result may be expressed in a power series by expanding the Bessel's Functions by equation (6), Art. 120 of Byerly's *Fourier's Series*, giving

$$J_{\frac{n-1}{2}}(B) = \frac{B^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}\left[1 - \frac{B^2}{2^2\left(\frac{n+1}{2}\right)}\right.$$

$$\left. + \frac{B^4}{2!2^4\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)} - \frac{B^6}{3!2^6\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} + \cdots\right] \tag{72}$$

$$J_{\frac{n+1}{2}}(B) = \frac{B^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}\Gamma\left(\frac{n+3}{2}\right)}\left[1 - \frac{B^2}{2^2\left(\frac{n+3}{2}\right)}\right.$$

$$\left. + \frac{B^4}{2!2^4\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)} - \frac{B^6}{3!2^6\left(\frac{n+3}{2}\right)\left(\frac{n+5}{2}\right)\left(\frac{n+7}{2}\right)} + \cdots\right] \tag{73}$$

Note that

$$\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = 2\frac{2\cdot4\cdot6\;\cdots n-2}{1\cdot3\cdot5\cdots n-1} \tag{74}$$

and

$$\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)} = 2\frac{2\cdot4\cdot6\cdots n}{1\cdot3\cdot5\cdots n+1} \tag{75}$$

Putting these values in equation (71) we obtain

$$\bar{p} = \frac{I^2}{c}\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!}\Big[-\frac{1}{n}+\frac{\sin^2 B}{2(n+1)}$$

$$+\cos B\frac{1}{n}\Big\{1-\frac{B^2}{2(n+1)}+\frac{B^4}{2!2^2}\frac{1}{(n+1)(n+3)}$$

$$-\frac{B^6}{3!2^3}\frac{1}{(n+1)(n+3)(n+5)}+\cdots\Big\}$$

$$+B\sin B\frac{1}{n}\Big\{\frac{1}{n+1}-\frac{B^2}{2}\frac{1}{(n+1)(n+3)}$$

$$+\frac{B^4}{2^2 2}\frac{1}{(n+1)(n+3)(n+5)}$$

$$-\frac{B^6}{2^3 3}\frac{1}{(n+1)(n+3)(n+5)(n+7)}+\cdots\Big\}\Big],$$

where

$$n = 2, 4, 6, \ldots \tag{76}$$

Equation (76) may be further improved for purposes of calculation by expanding the trigonometric functions in power series and collecting the terms. For this purpose

$$\frac{\sin^2 B}{2} = \frac{1-\cos 2B}{4} = \frac{B^2}{2!} - \frac{2^2B^4}{4!} + \frac{2^4B^6}{6!} - \frac{2^6B^8}{8!} + \cdots \tag{77}$$

$$\cos B = 1 - \frac{B^2}{2!} + \frac{B^4}{4!} - \frac{B^6}{6!} + \cdots \tag{78}$$

$$B\sin B = B^2 - \frac{B^4}{3!} + \frac{B^6}{5!} - \cdots \tag{79}$$

Equations (77), (78) and (79) substituted in (76) will give

$$\bar{p} = \frac{I^2}{c}\sum(-1)^{\frac{n}{2}}\frac{k^n}{n!}\cdot F_n(B) \tag{80}$$

where $F_n(B)$ is a polynomial in B^0, B^2, B^4, etc., where the coefficients of the several powers of B are contained in the table of page 464.

In this table the bottom row of terms gives the coefficients of the powers of B, when the summation indicated in (80) is performed with $n = 2, 4, 6 \ldots \infty$. The various terms in the columns were employed in obtaining the last row by addition.

The coefficient of B^{10} is not contained in the table, because of its numerous terms, but its value when summed up is

$$\frac{255n^4 + 6084n^3 + 51396n^2 + 177264n + 193536}{10!\,(n+1)(n+3)(n+5)(n+7)(n+9)}$$

Substituting the values of the coefficients multiplied by the corresponding powers of B and summing up as indicated in equation (80), we obtain for the power the expression

$$\begin{aligned}\bar{p} = \frac{I^2}{c}\Big[&k^2\left\{\frac{B^4}{60} - \frac{11B^6}{3780} + \frac{13B^8}{56700} - \frac{B^{10}}{93555} + \quad \cdots\right\} \\ &- k^4\left\{\frac{B^4}{1120} - \frac{B^6}{6480} + \frac{B^8}{83160} - \frac{B^{10}}{77395500} + \quad \cdots\right\} \\ &+ k^6\left\{\frac{B^4}{45360} - \frac{B^6}{21960960} + \frac{7B^8}{6!34720} - \quad \cdots\right\}\end{aligned} \qquad (81)$$

This equation gives the average power radiated in the aërial hemisphere from the flat-top of the antenna regarded as a separate radiator with the distribution that it has under the fundamental assumptions of the problem. The current is to be measured in absolute electrostatic units, and the power is in ergs per second.

In this equation $$B = \frac{2\pi b}{\lambda}$$

$$k = 2A = \frac{4\pi a}{\lambda}.$$

It remains to find how this power is modified by the mutual effect consisting of the interference between the waves emitted from the vertical portion of the antenna and the waves emitted from the horizontal part. This is the subject matter of Part III.

B^0	B^2	B^4	B^6	B^8
$-\dfrac{1}{n}$	$+\dfrac{1}{2!(n+1)}$	$-\dfrac{2^2}{4!(n+1)}$	$+\dfrac{2^4}{6!(n+1)}$	$-\dfrac{2^6}{8!(n+1)}$
$+\dfrac{1}{n}$	$-\dfrac{1}{2n(n+1)}$	$+\dfrac{1}{2!2^2n(n+1)(n+3)}$	$-\dfrac{1}{3!2^3n(n+1)(n+3)(n+5)}$	$+\dfrac{1}{4!2^4n(n+1)(n+3)(n+5)(n+7)}$
	$-\dfrac{1}{2!n}$	$+\dfrac{1}{2!2n(n+1)}$	$-\dfrac{1}{2^22!2!n(n+1)(n+3)}$	$+\dfrac{1}{2!3!2^3n(n+1)(n+3)(n+5)}$
		$+\dfrac{1}{4!n}$	$-\dfrac{1}{4!2n(n+1)}$	$+\dfrac{1}{2!2^24!n(n+1)(n+3)}$
			$-\dfrac{1}{6!n}$	$+\dfrac{1}{6!2n(n+1)}$
	$+\dfrac{1}{n(n+1)}$	$-\dfrac{1}{2n(n+1)(n+3)}$	$+\dfrac{1}{2!2^2n(n+1)(n+3)(n+5)}$	$+\dfrac{1}{8!n}$
		$-\dfrac{1}{3!n(n+1)}$	$+\dfrac{1}{3!2n(n+1)(n+3)}$	$-\dfrac{1}{3!2^3n(n+1)(n+3)(n+5)(n+7)}$
			$+\dfrac{1}{5!n(n+1)}$	$-\dfrac{1}{3!2!2^2n(n+1)(n+3)(n+5)}$
				$-\dfrac{1}{5!2n(n+1)(n+3)}$
				$-\dfrac{1}{7!n(n+1)}$
0	0	$-\dfrac{n+2}{4!2(n+1)(n+3)}$	$+\dfrac{3n^2+22n+32}{4!6(n+1)(n+3)(n+5)}$	$-\dfrac{3n^3+44n^2+196n+240}{5!2^4(n+1)(n+3)(n+5)(n+7)}$

PART III

THE MUTUAL TERM IN POWER DETERMINATION

94. The Trigonometric Relations.—In Art. **92,** equation (56), it has been shown that the power radiated through an element of surface consists of three terms in the form

$$dp = \frac{c}{4\pi}(E_\theta H_\phi + E_\psi H_\Sigma + 2\cos a\, E_\theta H_\Sigma)\, dS.$$

The first two of these terms we have already discussed. Putting in the values of E_θ and H_Σ from equations (19) and (55) the remaining power term, which we have for convenience called *mutual power,* becomes in the time average

$$\overline{dp} = \frac{I^2 \cos a\, dS}{\pi c r_0^2 \sin\theta \sin\psi} \sin\frac{Az}{r_0}\left\{\cos\psi \sin B - \sin(B\cos\psi)\right\}$$
$$\left\{\cos B \cos(A\cos\theta) - \sin B \cos\theta \sin(A\cos\theta) - \cos G\right\} \quad (82)$$

In forming this equation we have multiplied the expression for E_θ of eq. (19) by the expression for H_Σ, eq. (55). The product so obtained contains terms involving *sin* τ *cos* τ plus terms involving $\cos^2\tau$. The time average of the $\sin\tau\cos\tau$ terms is zero; while the time average of $\cos^2\tau$ is ½; these facts have been used in forming (82).

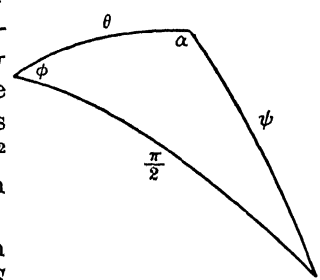

FIG. 10.

To be able to integrate equation (82) we must replace a, z, ψ and dS by their values in terms of θ, ϕ and r_0. By Fig. 3,

$$z = r_0 \cos\theta \quad (83)$$

$$dS = r_0^2 \sin\theta\, d\theta\, d\phi \quad (84)$$

In the spherical triangle of Fig. 10, a is represented, as defined, as the angle between θ and ψ, while opposite to a the side is $\pi/2$. The important trigonometric relation in a spherical triangle is as follows:

I. The cosine of any side is equal to the product of the cosines of the two other sides plus the continued product of the sines of these sides and the cosine of the included angle.

By this proposition, referring to Fig. 10, we see that

$$\cos \psi = \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta \cos \phi$$

$$= \sin \theta \cos \phi \tag{85}$$

By the same proposition

$$\cos \frac{\pi}{2} = \cos \theta \cos \psi + \sin \theta \sin \psi \cos \alpha;$$

$\therefore$

$$\cos \alpha = -\frac{\cos \theta \cos \psi}{\sin \theta \sin \psi} \tag{86}$$

or

$$\frac{\cos \alpha}{\sin \psi} = -\frac{\cos \theta \cos \psi}{\sin \theta \sin^2 \psi} \tag{87}$$

and by (85) this becomes

$$\frac{\cos \alpha}{\sin \psi} = -\frac{\cos \theta \cos \phi}{1 - \sin^2 \theta \cos^2 \phi} \tag{88}$$

95. Integration for Mutual Power.—Now substituting the trigonometric relations (83), (84), (85), (88) into equation (82), we obtain the following integral expression for the time average of the mutual power radiated through the *aërial hemisphere:*

$$\bar{p} = \frac{I^2}{c\pi} \int_0^{\pi/2} d\theta \sin (A \cos \theta) \left\{ \cos B \cos (A \cos \theta) - \right.$$

$$\left. \sin B \cos \theta \sin (A \cos \theta) - \cos G \right\}$$

$$\left[\cos \theta \int_0^{2\pi} \frac{\cos \phi \sin (B \sin \theta \cos \phi)\, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \right.$$

$$\left. - \cos \theta \sin \theta \sin B \int_0^{2\pi} \frac{\cos^2 \phi \, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \right] \tag{89}$$

This is a very complicated expression involving the integral of an integral.

We shall first proceed to perform the integration with respect to ϕ.

$$\text{Let } V = \int_0^{2\pi} \frac{\cos \phi \sin (B \sin \theta \cos \phi)\, d\phi}{1 - \sin^2 \theta \cos^2 \phi} \tag{90}$$

and break the integral into the sum of two integrals thus:

$$V = \int_0^{\pi} + \int_{\pi}^{2\pi}.$$

By a change of variable in the second of these two integrals by replacing ϕ by $\phi' + \pi$, we find that the integrand is unchanged, while the limits become 0 and π, so we may write

$$V = 2\int_0^\pi \frac{\cos\phi \sin(B \sin\theta \cos\phi)\, d\phi}{1 - \sin^2\theta \cos^2\phi} \tag{91}$$

Again decomposing this into the sum of two integrals we have

$$V = 2\left\{\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right\} \tag{92}$$

and changing the variable in the second integral by putting $\phi = \pi - \phi'$, the second integral becomes

$$\int_{\pi/2}^{\pi} = \int_{\pi/2}^{0} \frac{-d\phi'(-\cos\phi')(-\sin(B\sin\theta\cos\phi'))}{1 - \sin^2\theta\cos^2\phi'},$$

which by dropping the primes and substituting in (92) and (91) gives

$$V = 4\int_0^{\pi/2} \frac{\cos\phi \sin(B\sin\theta\cos\phi)\, d\phi}{1 - \sin^2\theta\cos^2\phi} \tag{93}$$

Now expanding in series as follows:

$$\sin(B\sin\theta\cos\phi) = B\sin\theta\cos\phi - \frac{B^3\sin^3\theta\cos^3\phi}{3!} + \frac{B^5\sin^5\theta\cos^5\phi}{5!} - \cdots,$$

and

$$\frac{1}{1 - \sin^2\theta\cos^2\phi} = 1 + \sin^2\theta\cos^2\phi + \sin^4\theta\cos^4\phi + \ldots \tag{93a}$$

and by taking the product of these two series we obtain

$$V = 4\int_0^{\pi/2} d\phi\Big[B\sin\theta\cos^2\phi + \left\{B - \frac{B^3}{3!}\right\}\sin^3\theta\cos^4\phi + \left\{B - \frac{B^3}{3!} + \frac{B^5}{5!}\right\}\sin^5\theta\cos^6\phi + \ldots\Big] \tag{94}$$

Integrating (94) by formula 483 of B. O. Peirce's Tables, we obtain

$$V = 2\pi \left[\frac{1}{2} B \sin \theta \right.$$
$$+ \frac{1 \cdot 3}{2 \cdot 4} \left\{ B - \frac{B^3}{3!} \right\} \sin^3 \theta$$
$$+ \frac{1 \cdot 3\ 5}{2 \cdot 4\ 6} \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} \right\} \sin^5 \theta$$
$$+ \frac{1\ 3\ 5\ 7}{2\ 4\ 6\ 8} \left\{ B - \frac{B^3}{3!} + \frac{B^5}{5!} - \frac{B^7}{7!} \right\} \sin^7 \theta$$
$$\ldots \tag{95}$$

We shall next proceed to perform the second integration with respect to ϕ indicated in (89). For abbreviation let us write

$$W = \int_0^{2\pi} \frac{\cos^2 \phi \, d\phi}{1 - \sin^2 \theta \cos^2 \phi} = 4 \int_0^{\pi/2} \frac{\cos^2 \phi \, d\phi}{1 - \sin^2 \theta \cos^2 \phi}$$

by reasoning similar to the above. Expanding the denominator by (93*a*), we have

$$W = 4 \int_0^{\pi/2} d\phi \cos^2\phi \left\{ 1 + \sin^2 \theta \cos^2 \phi + \sin^4 \theta \cos^4 \phi + \ldots \right\}$$
$$= 2\pi \left\{ \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \sin^2 \theta + \frac{1 \cdot 3\ 5}{2 \cdot 4 \cdot 6} \sin^4 \theta + \ldots \right\} \tag{96}$$

(If we need it, this integral can be obtained by direct integration in the form

$$W = 2\pi \left\{ \frac{1}{\cos \theta \, (1 + \cos \theta)} \right\}$$

but the expanded form is more useful for our purpose.)

Now substituting (95) and (96) in (89) we obtain

$$\bar{p} = \frac{2I^2}{c} \int_0^{\pi/2} d\theta \cos \theta \sin (A \cos \theta) \left\{ \cos B \cos (A \cos \theta) \right.$$
$$\left. - \sin B \cos \theta \sin (A \cos \theta) - \cos G \right\}$$
$$\left[\frac{1}{2} (B - \sin B) \sin \theta \right.$$
$$+ \frac{1 \cdot 3}{2 \cdot 4} \left(B - \frac{B^3}{3!} - \sin B \right) \sin^3 \theta$$
$$+ \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B \right) \sin^5 \theta$$
$$\left. + \ldots \right] \tag{97}$$

To evaluate this expression we must obtain the following integrals:

$$I_1 = \int_0^{\pi/2} d\theta \sin^n \theta \cos \theta \frac{\sin (2A \cos \theta)}{2} \tag{98}$$

$$I_2 = \int_0^{\pi/2} d\theta \sin^n \theta \cos^2 \theta \sin^2 (A \cos \theta) \tag{99}$$

$$I_3 = \int_0^{\pi/2} d\theta \sin^n \theta \cos \theta \sin (A \cos \theta) \tag{100}$$

where $n = 1, 3, 5, 7, \ldots$

I_3 is the simplest of these integrals and will be considered first. By expanding $\sin (A \cos \theta)$ in series we have

$$I_3 = \int_0^{\pi/2} d\theta \sin^n \theta \left\{ A \cos^2 \theta - \frac{A^3 \cos^4 \theta}{3!} + \frac{A^5 \cos^6 \theta}{5!} - \ldots \right\}$$

which by Byerly Int. Calc., Art. 99, Ex. 2, may be integrated in Gamma Functions as follows:

$$I_3 = A \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}+1\right)} - \frac{A^3}{3!}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+4}{2}+1\right)} + \frac{A^5}{5!}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+6}{2}+1\right)} - \frac{A^7}{7!}\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+8}{2}+1\right)} + \ldots \tag{101}$$

If we note that

$$\Gamma\left(\frac{n+2}{2}+1\right) = \frac{n+2}{2}\frac{n}{2}\Gamma\left(\frac{n}{2}\right)$$

$$\Gamma\left(\frac{n+4}{2}+1\right) = \frac{n+4}{2}\frac{n+2}{2}\frac{n}{2}\Gamma\left(\frac{n}{2}\right)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right),$$

we obtain

$$I_3 = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{A}{n(n+1)} - \frac{A^3}{3!}\,\frac{3\ 1}{n(n+2)(n+4)}\right.$$

$$\left. + \frac{A^5}{5!}\,\frac{5\cdot 3\cdot 1}{n(n+2)(n+4)(n+6)} - \cdots\right].$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\,\frac{A}{n(n+2)}\left\{1 - \frac{A^2}{2(n+4)} + \right.$$

$$\left.\frac{A^4}{4\cdot 2\,(n+4)(n+6)} - \frac{A^6}{6\cdot 4\cdot 2(n+4)(n+6)(n+8)} + \cdot\ \cdot\right\} \quad (102)$$

In like manner

$$I_1 = \frac{2A}{2n(n+2)}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left\{1 - \frac{(2A)^2}{2(n+4)} + \right.$$

$$\left.\frac{(2A)^4}{4\ 2(n+4)(n+6)} - \frac{(2A)^6}{6\ 4\cdot 2(n+4)(n+6)(n+8)} + \cdot\ \cdot\right\} \quad (103)$$

Now taking up integral I_2 from equation (99), let us write it

$$I_2 = \int_0^{\pi/2} d\theta \sin^n\theta\cos^2\theta\left\{\frac{1-\cos(2A\cos\theta)}{2}\right\},$$

and expanding $\cos(2A\cos\theta)$ in series, obtain

$$I_2 = \frac{1}{2}\int d\theta\left[\sin^n\theta\cos^2\theta\left\{\frac{(2A)^2\cos^2\theta}{2!} - \frac{(2a)^4\cos^4\theta}{4!} + \cdot\ \cdot\ \cdot\right\}\right.$$

This equation, integrated in Gamma Functions between the limits 0 and $\pi/2$ gives

$$I_2 = \frac{1}{2}\left[\frac{(2A)^2}{2!}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{n+4}{2}+1\right)}\right.$$

$$\left. - \frac{(2A)^4}{4!}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{n+6}{2}+1\right)} + \cdot\ \cdot\ \cdot\right],$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{2A^2}{n(n+2)(n+4)} \left\{\frac{3}{2} - \frac{5(2A)^2}{4\cdot 2(n+6)} + \frac{7(2A)^4}{6\ 4\ 2(n+6)(n+8)} - \cdot\right\} \tag{104}$$

Employing the values of I_1, I_2, I_3 found in equations (103), (104) and (102) we may write the expression for the mutual power in the integrated form

$$p = \frac{2I^2}{c}\Big[\cos B \Big\{(B - \sin B)\frac{A}{3!}\Big[1 - \frac{(2A)^2}{2\cdot 5} + \frac{(2A)^4}{4\cdot 2\ 5\cdot 7} - \frac{(2A)^6}{6\cdot 4\cdot 2\cdot 5\cdot 7\cdot 9} + \ldots\Big]$$

$$+ \left(B - \frac{B^3}{3!} - \sin B\right)\frac{3!A}{5!}\Big[1 - \frac{(2A)^2}{2\ 7} + \frac{(2A)^4}{4\ 2\cdot 7\cdot 9} - \frac{(2A)^6}{6\cdot 4\cdot 2\cdot 7\cdot 9\cdot 11} + \ \cdot\ \cdot\Big]$$

$$+ \left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{5!A}{7!}\Big[1 - \frac{(2A)^2}{2\cdot 9} + \frac{(2A)^4}{4\cdot 2\cdot 9\cdot 11} - \frac{(2A)^6}{6\cdot 4\cdot 2\cdot 9\cdot 11\cdot 13} + \ \cdot\ \cdot\Big]$$

$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\Big\}$$

$$- \sin B\Big\{\frac{1}{2}\left(B - \sin B\right)\frac{2A^2}{3\cdot 5}\Big[\frac{3}{2} - \frac{5(2A)^2}{4\cdot 2\cdot 7} + \frac{7(2A)^4}{6\cdot 4\cdot 2\cdot 7\cdot 9} - \frac{9(2A)^6}{8\cdot 6\cdot 4\cdot 7\cdot 9\cdot 11} + \ \cdot\ \cdot\ \cdot\Big]$$

$$+ \frac{1\cdot 3}{2\cdot 4}\left(B - \frac{B^3}{3!} - \sin B\right)\frac{2A^2}{3\cdot 5\cdot 7}\Big[\frac{3}{2} - \frac{5(2A)^2}{4\cdot 2\cdot 9} + \frac{(2A)^4}{6\cdot 4\cdot 2\cdot 9\ 11} - \frac{(2A)^6}{8\cdot 6\cdot 4\cdot 2\cdot 9\cdot 11\cdot 13} + \quad\Big]$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \sin B\right)\frac{8A^2}{3\cdot 5\cdot 7\cdot 9}\Big[\frac{3}{2} - \frac{5(2A)^2}{4\cdot 2\cdot 11} + \frac{(2A)^4}{6\cdot 4\cdot 2\cdot 11\cdot 13} - \frac{9(2A)^6}{8\cdot 6\cdot 4\cdot 2\cdot 11\cdot 13\cdot 15} + \ \cdot\ \cdot\ \cdot\Big]$$

$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\Big\}$$

$$- \cos G\Big\{\frac{1}{2}\left(B - \sin B\right)\frac{A}{3}\Big[1 - \frac{A^2}{2\cdot 5} + \frac{A^4}{4\cdot 2\cdot 5\cdot 7} - \frac{A^6}{6\cdot 4\cdot 2\cdot 5\cdot 7\cdot 9} + \ \cdot\ \cdot\ \cdot\Big]$$

$$+\frac{1\cdot 3}{2\cdot 4}\left(B-\frac{B^3}{3!}-\sin B\right)\frac{2A}{3\cdot 5}\left[1-\frac{A^2}{2\cdot 7}+\frac{A^4}{4\cdot 2\cdot 7\cdot 9}-\frac{A^6}{6\cdot 4\cdot 2\cdot 7\cdot 9\cdot 11}+\ldots\right]$$

$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(B-\frac{B^3}{3!}+\frac{B^5}{5!}-\sin B\right)\frac{8A}{3\cdot 5\cdot 7}\left[1-\frac{A^2}{2\cdot 9}+\frac{A^4}{4\cdot 2\cdot 9\cdot 11}-\frac{A^6}{6\cdot 4\cdot 2\cdot 9\cdot 11\cdot 13}+\ldots\right]$$

$$+\ldots\ldots\ldots\Big\}\Big] \qquad (104)$$

If now we recall that $G = A + B$, it will be seen that the equation (104) is entirely in terms of A and B and I.

For purpose of computation it is found advisable to expand all of the trigonometrical expressions in power series and then perform with them the indicated operations. This was done with considerable labor and gave the following expression for mutual power:

$$\bar{p} = \frac{2I^2}{c}\Big[A^2\left\{.0261B^4 - .00586B^6 + .000515B^8 - \ldots\right\}$$

$$+ A^3\left\{.0055B^3 - .00317B^5 + .000442B^7 - .0000297B^9 + \ldots\right\}$$

$$+ A^4\left\{-.00343B^4 + .000808B^6 - \text{[illegible]}B^8 + \ldots\right\}$$

$$+ A^5\left\{-.00106B^3 + .000603B^5 - .0000828B^7 + .0000055B^9 - \ldots\right\}$$

$$+ A^6\left\{.00126B^4 - .00028B^6 + .0000238B^8 - \ldots\right\}\Big] \qquad (105)$$

This equation gives the time average of the power radiated in the aërial hemisphere by the mutual effect of the fields from both parts of the antenna and is the correction to be added to the power radiated by the two parts, estimated as independent of each other. The current I is in absolute c.g.s. electrostatic units, and the power is in ergs per second.

96. Summation of Flat-top Power and Mutual Power.—We have obtained in equation (81) the time average of flat-top radiated power, and in equation (105) the time average of mutual radiated power. If we replace the k of (81) by its value in

terms of A, the two expressions may be added together. At the time of the addition we shall reduce the units to the practical system by multiplying the right-hand sides of both power equations by 30 times the velocity of light in centimeters per second (*i.e.*, by 30 c), and obtain

$$
\begin{aligned}
p = 60I^2 \Big[& A^2 \{ 0595B^4 - .01167B^6 + .000974B^8 - \\
& \qquad\qquad .0000458B^{10} + \ldots \} \\
& + A^3 \{ .0055B^3 - .00317B^5 + .000442B^7 - \\
& \qquad\qquad .0000297B^9 + \ldots \} \\
& - A^4 \{ .01058B^4 - .00204B^6 + .000171B^8 + \\
& \qquad\qquad .0000082B^{10} + \ldots \} \\
& - A^5 \{ .00106B^3 - .000603B^5 + .0000828B^7 - \\
& \qquad\qquad 0000055B^9 + \ldots \} \\
& + A^6 \{ .00196B^4 - .00032B^6 + .000033B^8 - \ldots \} \\
& + \ldots \Big]
\end{aligned}
\tag{106}
$$

This is the total power contribution of the flat top by virtue of its individual and mutual action. The power is in watts, and the current I is in amperes.

Certain Tables computed in the next Part of this chapter make calculations with this series comparatively simple.

IV. COMPUTATIONS OF RADIATION RESISTANCE

97. Equation for Radiation Resistance.—If

a = length of vertical part in meters,
b = length of horizontal part in meters,
λ_0 = the natural wavelength of the antenna in meters,
λ = the wavelength in meters of the antenna as loaded with inductance at its base,

$$A = \frac{2\pi a}{\lambda},$$

$$B = \frac{2\pi b}{\lambda},$$

$$q = \frac{\pi\lambda_0}{\lambda}.$$

we may obtain the radiation resistance of the antenna by dividing the power radiated by the mean square of the current at the base of the antenna This mean square current at the base of the antenna is by (5)

$$\overline{I_0^2} = \frac{I^2 \sin^2 (q/2)}{2}$$

Performing this division as to the flat-top power employing equation (106) and adding the result to the radiation resistance for the vertical portion as given in equation (44) we obtain for the total radiation resistance of the antenna the equation

$$R_\Omega = \frac{1}{\sin^2 (q/2)} \Big\{ R_1 - R_2 \cos q - R_3 \sin q + r_2 A^2 + r_3 A^3 - r_4 A^4 - r_5 A^5 + r_6 A^6 + \ldots \Big\} \qquad (107)$$

This is Radiation Resistance in Ohms, where

$$R_1 = 15 \left\{ \frac{2+2}{3!2}(2A)^2 - \frac{4+2}{5!4}(2A)^4 + \frac{6+2}{7!6}(2A)^6 - \ldots \right\}$$

$$R_2 = 15 \left\{ \frac{2^2+2^2-4}{3!2}(2A)^2 - \frac{4^2+2^4-6}{5!4}(2A)^4 + \frac{6^2+2^6-8}{7!6}(2A)^6 - \ldots \right\}$$

$$R_3 = 15 \left\{ \frac{3^2+2^3-5}{4!3}(2A)^3 - \frac{5^2+2^5-7}{6!5}(2A)^5 + \frac{7^2+2^7-9}{8!7}(2A)^7 - \ldots \right\}$$

$$r_2 = 120 \left\{ .0595B^4 - .01167B^6 + .000974B^8 - .0000158B^{10} + \ldots \right\}$$

$$r_3 = 120 \left\{ .0055B^3 - .00317B^5 + .000442B^7 - .0000297B^9 + \ldots \right\}$$

$$r_4 = 120 \left\{ .0106B^4 - .00204B^6 + .000171B^8 - .0000082B^{10} + \ldots \right\}$$

$$r_5 = 120 \left\{ .00106B^3 - .000602B^5 + .000083B^7 - .0000055B \cdot \ldots \right\}$$

$$r_6 = 120 \left\{ .00196B^4 - .00032B^6 + .000033B^8 - \ldots \right\} \qquad (108)$$

98. Tables of Coefficients of Radiation Resistance.—There follow in Tables I and II the values of the coefficients R_1, R_2, R_3,

r_2, r_3, r_4 r_5, r_6, foi various values of A and B respectively. These tables have been computed by the equations (108).

Table I.—Coefficients R_1, R_2, and R_3

$2A$	$\lambda/4a$	R_1	R_2	R_3
0 1	31 416	0 04998	0 049919	0 002498
0 2	15 70	0 19971	0 19870	0 01994
0 3	10 47	0 44848	0 44344	0 06700
0 4	7 85	0 79521	0 78107	0 1579
0 5	6 28	1 2383	1 20634	0 3060
0 6	5 236	1 7759	1 6969	0 5241
0 7	4 488	2.4055	2 2602	0 8232
0 8	3 927	3 1240	2 8786	1 2137
0 9	3.491	3.9290	3 5403	1 696
1 0	3 141	4 8165	4 2315	2 300
1 1	2 854	5 7837	4 9383	3 009
1 2	2.616	6 8232	5.6442	3 823
1.4	2 241	9 150	7 000	5.90
1 5	2 092	10 3392	7.611	6 999
1.6	1.962	11 64	8 15	8 35
1 732	1 812	13 415	8 798	10 113
1 8	1 743	14.40	9.10	11 20
2.00	1 570	17.241	9.550	14 354
2.20	1.427	20 15	9.55	17 80
2.236	1.403	20.778	9 508	18 470
2.40	1 307	23 22	9 00	21 42
2 60	1 207	26 37	7.90	25.20
2.642	1.189	27.053	7 60	25 927
2.80	1.121	29.40	6 22	29 05
3 141	1.000	34.45	2.12	35.64

Table II.—Coefficients r_2, r_3, etc.

B	$\lambda/4b$	r_2	r_3	r_4	r_5	r_6
1.4	1.112	18.36	0.282	2.34	0.047	0 806
1.2	1.31	11.09	0.370	2 00	0 079	0 409
1.0	1.57	5.85	0.330	1.05	0.054	0.211
0.8	1.96	2.48	0.209	0.459	0 038	0.090
0.6	2.61	0.858	0 065	0.152	0.022	0.0362
0.4	3.93	0.177	0.042	0.032	0.0074	0.0062
0.2	7 85	0.0092	0.005	0.002	0.001	0 0004
0.37	4.23	0.130	0.019	0.0232	0.0060	0.0043
0.57	2.75	0.703	0.101	0.125	0.0194	0.0127
0.77	2 04	2.234	0.218	0.400	0.040	0.0752
0.97	1.62	5.260	0.317	0 937	0.061	0.180
1.17	1.34	10.18	0.367	1.822	0.073	0.356
1.37	1.15	17.20	0.280	2.990	0.059	0.504

99. Curves of Resistance Due to Radiation from the Flat-top.—We shall now proceed to discuss the curves of radiation resistance of variously proportioned antennæ when employed at various wavelengths relative to the natural wavelength. As pre-

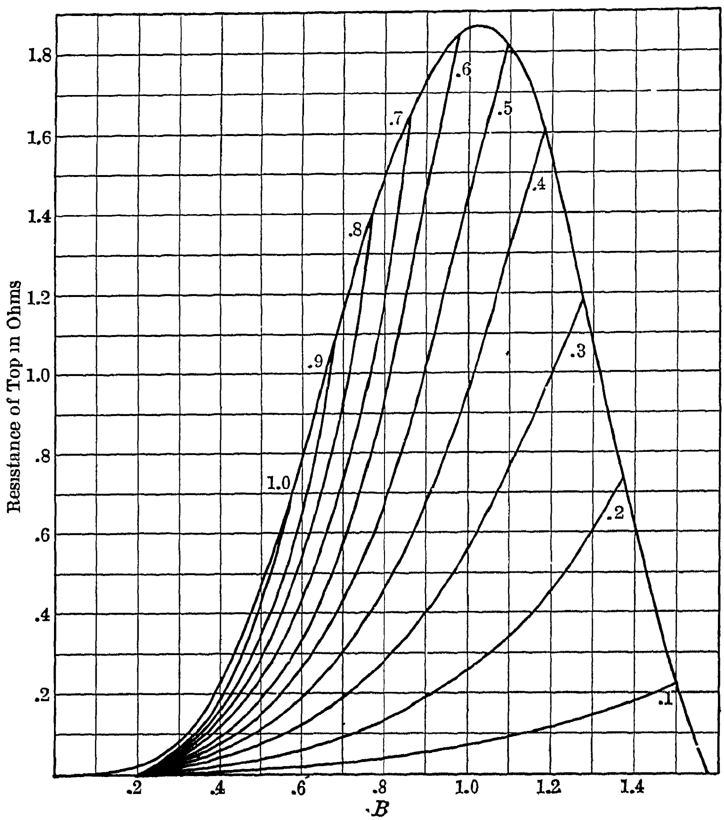

FIG. 11.—Radiation Resistance of horizontal top portion of antenna plotted against values of B. The separate curves numbered .1, .2, .3, etc. to 1.0 are for values of A = .1, .2, .3, etc. to 1.0.

liminary, the resistance due to radiation from the flat-topped portion of the antennæ is first computed. The equation for this is the summation of terms in (107) containing the small r's as factors; that is,

$$R_\Omega = \frac{1}{\sin^2 (q/2)}\left\{r_2A^2 + r_3A^3 - r_4A^4 - r_5A^5 + r_6A^6 + \dots\right\}$$

due to (109)

flat-top
in which

$$A = \frac{2\pi a}{\lambda}$$

$$B = \frac{2\pi b}{\lambda}$$

$$q = \frac{\pi\lambda_0}{\lambda} = 2(A + B).$$

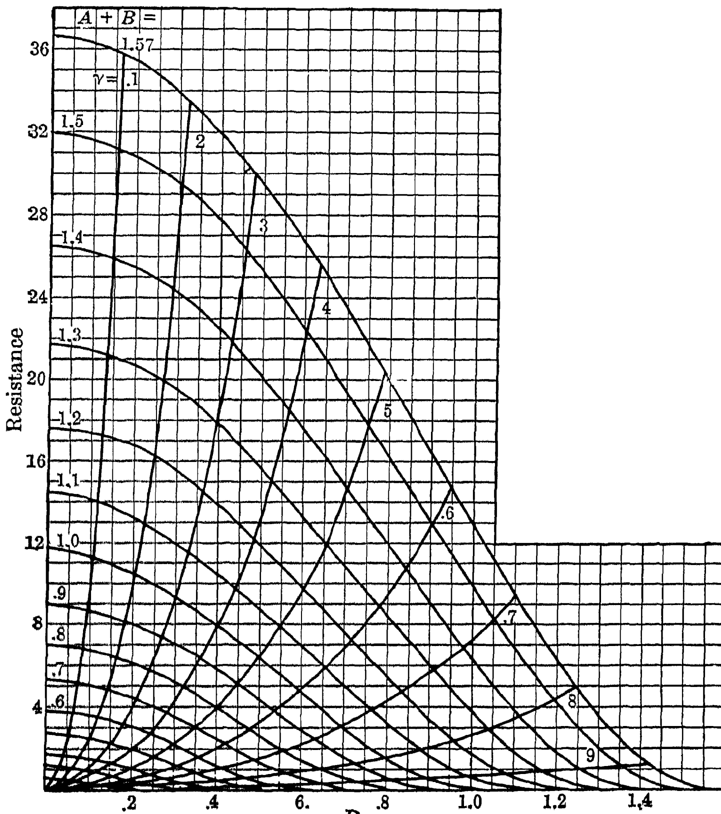

FIG. 12.—Total Radiation Resistance plotted against values of B. The separate curves through the origin are for designated values of γ. Separate curves not passing through origin are for different values of $A + B$.

Since the coefficients (small r's) are functions of B only, as given in Table II, it follows that when A and B are given, the value of the flat-top R may be computed. The results of the computations for various values of A and B are plotted in Fig. 11.

In this figure values of B are the abscissæ, while the flat-top resistances in ohms are ordinates. The separate curves numbered .1, .2, .3, etc., to 1.0 are for values of $A = 0.1, 0.2, 0.3,$ etc. to 1.0.

The outside end-points of these several curves, through which a limiting curve is drawn, are determined by the equality of the

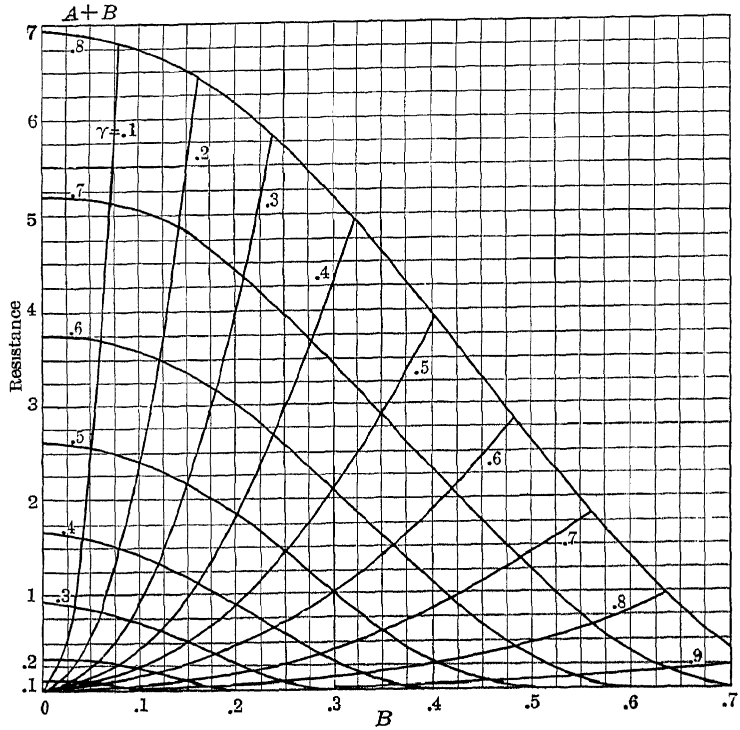

FIG. 13.—Enlarged view of some of the curves of Fig. 12.

impressed wavelength λ and the natural wavelength of the antenna λ_0; that is, by the value of $A + B = \pi/2$, which is the largest value $A + B$ can have for the fundamental oscillation of the antenna.

100. Curves of Total Radiation Resistance.—The next step consists in computing the radiation resistance of the vertical portion of the antenna, using the first three terms of equation (107), and employing a large number of values of A and B. To these values of resistance due to the vertical portion of the antenna the corresponding resistance of the flat-top are added so

as to give the total resistance of the antenna for various values of A and B. Curves of resistance for various values of $A + B$ are then plotted in Fig. 12, with values of B as abscissæ and values of resistance as ordinates. Figure 13 is an enlarged view of some of the curves that are on too small a scale to read in Fig. 12. Then to make the family of curves more useful for ready reference a series of curves are drawn through all the points which have a common ratio of length of flat-top to length of total antenna. This ratio is designated by γ, where

$$\gamma = \frac{B}{A + B} = \frac{b}{a + b} \tag{110}$$

with

b = length of flat-top

a = length of vertical part of antenna.

These γ-curves all pass through the origin.

Next as a final step the curves of Fig 14 are taken from the curves of Figs. 12 and 13 with the new set

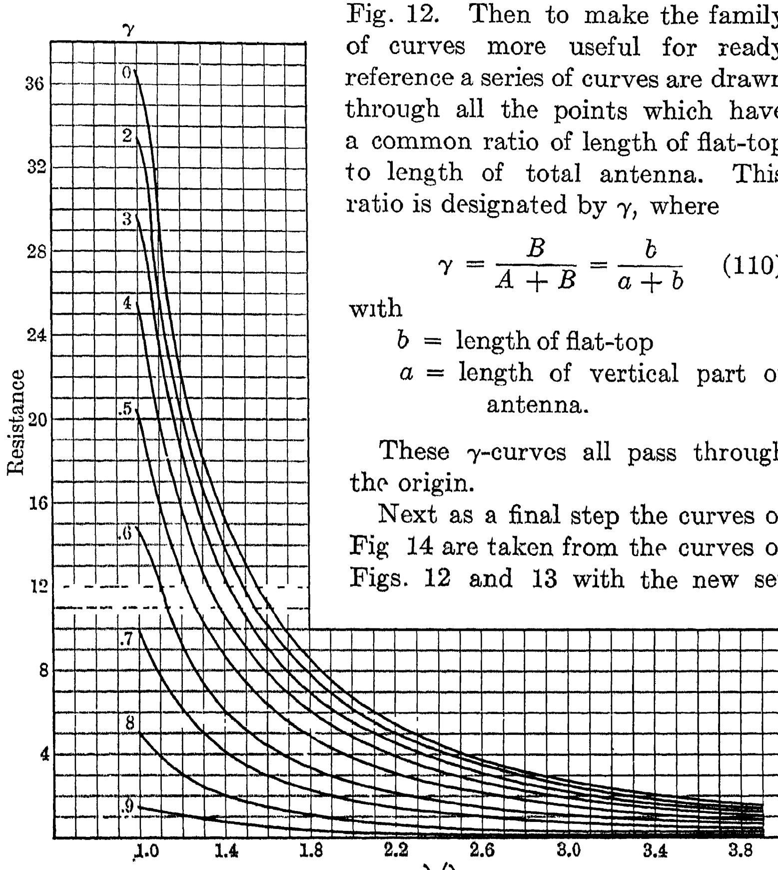

FIG. 14.—Total Radiation Resistance plotted against λ/λ_0. The separate curves marked 0, .2, .3, etc. are for values of γ = 0, 0.2, 0.3, etc.

of parameters. These curves of Fig. 14 are the final curves of total radiation resistance, and are in terms of the ratio of the wavelength employed to the natural wavelength (that is λ/λ_0) and the ratio of the length of flat-top to total length of antenna (that is γ). Fig. 15 is merely a magnified view of certain of the curves that are too small to read on Fig. 14.

101. Total Radiation Resistance of a Straight Vertical Antenna at Various Wavelengths Obtained by Inductance at the Base.—As an example, let it be required to find the total radiation resistance of a straight vertical antenna for various wavelengths obtained by adding various inductances at the base. For this case $\gamma = 0$, and from the $\gamma = 0$ curve of Figs. 14

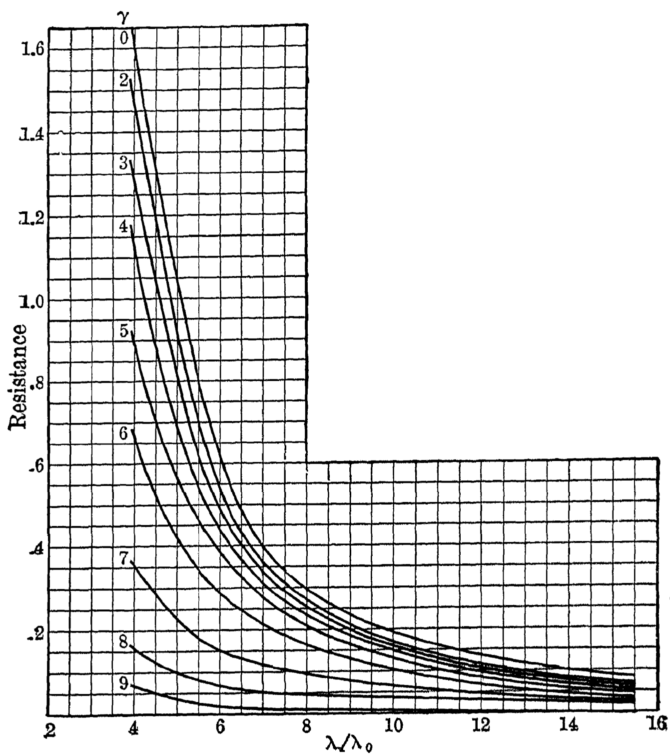

FIG. 15.—Magnified view of some of the curves of Fig. 14 with the larger values of λ/λ_0.

and 15 R may be directly read. The values which were used in plotting this curve are given in Table III, where they are compared with the corresponding values computed on the assumption that the oscillator is a Hertzian doublet. This latter assumption[1]

[1] This result is obtained by taking equation (53) of Art. 78, and noting that the power is radiated only in the upper hemisphere, whence

$$R = \frac{40\pi^2 l^2}{\lambda^2} \text{ ohms;}$$

Table III.—Resistance of a Straight Vertical Antenna for Different Values of Wavelength Obtained by Inductance at the Base

λ/λ_0 ratio of wavelength to natural wavelength	R, radiation resistance in ohms computed by present theory	Radiation resistance in ohms computed on doublet theory
1 00	36 57	98 7
1 12	26 40	78 7
1 21	21 70	67 3
1 31	17 65	57 5
1 43	14 28	48 2
1 57	11 62	40 0
1 74	9 10	32 6
1 97	6 92	25 4
2 24	5 19	19 7
2.62	3 78	14 4
3 14	2 58	10 0
3 93	1 65	6 40
5 26	0 90	3 60
7.85	0 30	1 16
15.70	0 082	0 40
31.42	0 01	0 10

gives

$$R = 160 \frac{\pi^2 a^2}{\lambda^2}.$$

It is seen that the departure of the present theory from the doublet theory is very large for the straight vertical antenna, as should be expected.

It should be noted that the first value in the column of resistances computed by the present theory agrees with the value for this case computed by Abraham in the work cited in Art. **89.** This one value, for the fundamental oscillation, is the only value arrived at by Abraham and is the case of a straight vertical antenna oscillating with its natural frequency. Abraham's other computed values are for the harmonic vibrations with more than one loop of potential always without loading the antenna by inductance, and without any flat-top extension of the antenna.

For convenience Table II at the end of the book contains computed values of Total Radiation Resistance for Flat-top

but l is length of whole doublet, and therefore is $2a$, whence

$$R = 160 \frac{\pi^2 a^2}{\lambda^2}$$

Antennæ of various ratios of horizontal length to vertical length and for various ratios of wavelength λ to natural wavelength λ_o.

102. Comparison of Computations on the Present Theory with Dr. Austin's Values for the Battleship "Maine."—Figure 16 gives the Radiation Resistance of the Antenna of the Battleship "Maine" as computed by the present Theory in comparison with Dr. Austin's measured values of the total resistance of this antenna, and in comparison with values computed on the doublet

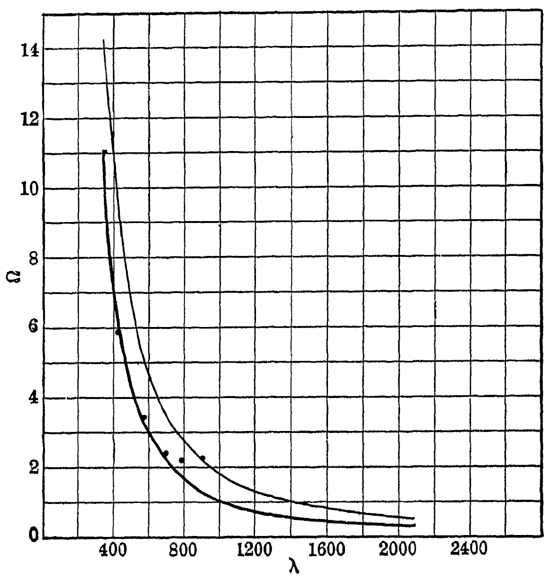

FIG. 16.—Total Radiation Resistance *vs.* Wave length for the Antenna of the Battleship "Maine." Black dots are Dr. Austin's observed values; heavy line, computations by present theory; light line, computations by doublet theory.

theory of Hertz. The black dots of Fig. 16 are Dr. Austin's observed values. The heavy line was obtained by computation by the present theory, and the weaker line, by computation regarding the antenna as a doublet of half-length equal to the vertical height of the antenna.

It is seen that the departure between the present theory and the doublet theory is not so great as in the case of the straight vertical antenna, for the reason that the doublet theory becomes more nearly correct as the half-length of the oscillator becomes small in comparison with the wavelength.

Neither of the theories gives a rising value of the resistance with increase of wavelength, and, as Dr. Austin has pointed out, his rising values for long waves are probably not due to radiation from the antenna but possibly to dielectric hysteresis in the ground beneath the flat-top.

I do not give more extended comparisons with experimental values at the present time, because I am now making some experiments to see how much reliance may be placed in antenna resistance measurements made by buzzer methods of excitation in comparison with measurements made by excitation with gaseous oscillators and other methods of continuous excitation.

103. Example of Different Methods of Constructing an Antenna that Will Have a Specified Resistance for a Given Wavelength.—Let it be required to construct an antenna that will have a given resistance (4 ohms, say) for a given wavelength (2000 meters, say). To solve this problem, it is only necessary to look up the four ohm point on the different γ-curves of Figs. 14 or 15, and find the value of λ/λ_0. We can then find the λ_0 of the antenna, since λ is given. Dividing the λ_0 by 4 we obtain the total length of antenna. The value of γ gives the fractional part of this length which is to be horizontal. The complete result is tabulated in Table IV.

Table IV.—Constants of the Different Antennæ that have 4 Ohms Resistance at 2000 Meters

γ	λ/λ_0	λ_0	Total length, meters	Vertical length, meters	Horizontal length, meters	Intensity factor in horizontal plane
0.8	1.075	1860	465	93 0	372.0	0.275
0.7	1.39	1435	359	107 7	251.3	0 300
0.6	1.67	1198	299	119.6	179.4	0.310
0.5	1.94	1030	258	129.0	129 0	0.312
0.4	2.18	916	229	137.4	91.6	0.313
0.3	2.32	861	215	150.5	64 5	0.314
0.2	2.44	820	205	164.0	41.0	0.315
0.0	2.52	793	198	198.0	00.0	0.320

The question as to which of these antenna to choose for the given purpose is now chiefly a problem in economics. The economic question is, which, for example, is cheaper: Two poles or towers 93 meters high and 372 meters apart, or one tower

198 meters high? This of course pre-supposes that it is designed to use a flat-top antenna instead of some other type, such as an umbrella.

The problem is, however, not wholly economic because the lower antenna would be preferable as a receiving antenna on account of its weaker response to atmospheric disturbances. There is also the further question as to which of the tabulated antennæ will give the greatest vertical intensity of electric and magnetic force on the horizon at a distant receiving station. This is the subject matter of the next Part (Part V).

PART V

FIELD INTENSITIES AND SUMMARY

104. The Electric and Magnetic Intensities at a Distant Point in the Horizontal Plane.—Equation (19) gives the values of the electric and magnetic intensities at a distant point due to the vertical portion of the antenna. If we replace I of that equation by its value in terms of I_0 from equation (6), and make $\cos\theta = 0$, we have the intensities in the horizontal plane in terms of I_0, which is the amplitude of the current at the base of the antenna. This gives

$$E_\theta = H_\phi = \frac{2I_0}{cr_0}\cos\frac{2\pi}{\lambda}(ct - r_0)\left[\frac{\cos B - \cos G}{\sin\frac{\pi\lambda_0}{2\lambda}}\right] \qquad (111)$$

The quantities outside the square brackets are constant for a given distance r_0 and a given amplitude of transmitting current I_0. The *relative intensities* are therefore determined by the factor in the square brackets, which we may designate by

$$X = \frac{\cos B - \cos G}{\sin\frac{\pi\lambda_0}{2\lambda}} \qquad (112)$$

Using the values of B, G, given in equation (20) and the value of γ in (110), this equation (112) becomes

$$X = \frac{\cos\gamma\left(\frac{\pi\lambda_0}{2\lambda}\right) - \cos\frac{\pi\lambda_0}{2\lambda}}{\sin\frac{\pi\lambda_0}{2\lambda}} \qquad (113)$$

This quantity X we shall call "The Intensity Factor in the Horizontal Plane." It is to be noted that the electric and magnetic intensities in the horizon plane are not effected by radiation from the flat-top; for, by equation (55), the field intensities from the flat-top are zero for $z = 0$; that is, all over the horizontal plane through the origin.

In Fig. 17 the Intensity Factor in the Horizontal Plane is plotted for various values of γ and various values of λ/λ_0. Taking from these curves the values of the intensity factors corresponding to the values of γ and λ/λ_0 of Table IV we obtain the results in the last column of Table IV. It is seen that the intensity factor is slightly smaller for the larger values of the relative length of flat-top. This diminished value of the intensity factor should be compensated by the use of a slightly larger transmitting current. The required increase of current may be easily computed by equation (111).

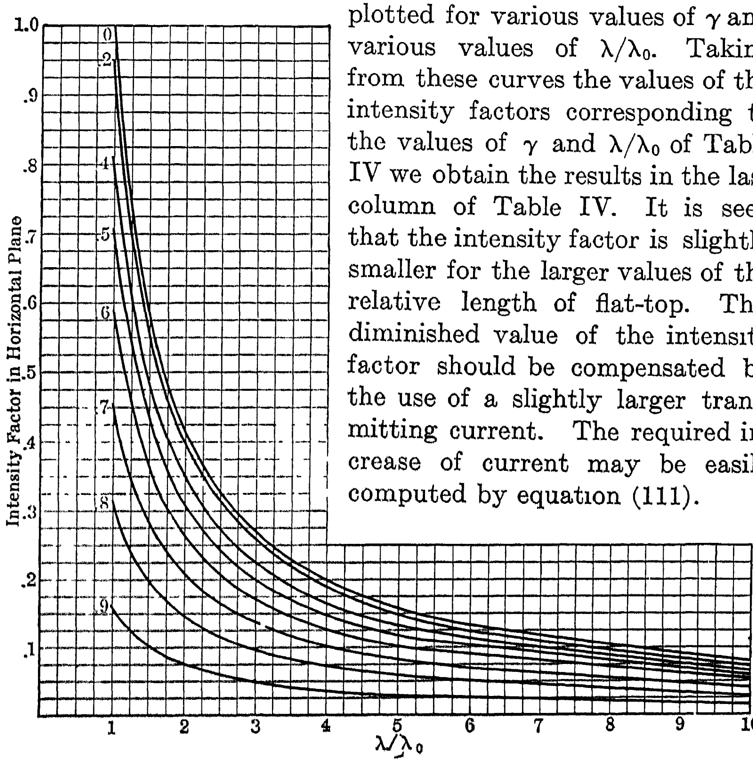

FIG. 17.—Relative intensity of the vertical component of Electric Force in a horizontal plane at a given distance from various antennæ and for a given amplitude of transmitting current.

105. Summary.—This chapter contains a mathematical theory of the flat-top antenna. The process employed consists in the integration of the effects of an aggregate of doublets assumed to be distributed along the antenna so as to give a current distribution described by equation (1) and illustrated in Fig. 2. The electric and magnetic field intensity due to each of the doublets is determined by the Maxwell and Hertz Theories for

all distant points in space. These field intensities are summed up for all the doublets with strict allowance for the differences of phase due to different doublets; the summation gives the resultant field intensities. Then by Poynting's theorem the power radiated from the antenna through a distant hemisphere (bounded by the earth's surface assumed plane) is computed by the integration of a number of intricate expressions. From the radiated power the radiation resistance is obtained by dividing by the mean square of the current at the base of the antenna Tables of coefficients for computing radiation resistance are given, and curves are plotted of the calculated values of radiation resistance for different ratios of the length of the flat-top to the total length of the antenna and for different relative wavelengths obtained by loading the antenna with inductance. Table II at end of volume gives for ready reference computed values of Radiation Resistance for Various Antennæ used at various wavelengths. Curves are also given for determining the relative electric and magnetic field intensities in the horizontal plane for differently proportioned antennæ variously loaded. Various equations developed in the treatment may find application to problems in the design of radiotelegraphic stations. Although this investigation was undertaken in ignorance of a simple case investigated by Professor Max Abraham, by a similar fundamental method, his work was discovered early in the course of the treatment and served as a check on one of the resistance values here given.

APPENDIX AND TABLES

APPENDIX I

MATHEMATICAL NOTES

Note 1. Proof that the Sum of Two or More Solutions of a Homogeneous Linear Differential Equation is a Solution.—Let us take for example the equation

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} \tag{1}$$

Suppose that

$$i = i_1 \text{ is a solution} \tag{2}$$

and

$$i = i_2 \text{ is another solution} \tag{3}$$

to prove that $i_1 + i_2$ is a solution.

By condition (2), i_1 substituted for i in equation (1) reduces the right hand to zero: that is

$$0 = L\frac{d^2i_1}{dt^2} + R\frac{di_1}{dt} + \frac{i_1}{C} \tag{4}$$

Likewise, condition (3) gives

$$0 = L\frac{d^2i_2}{dt^2} + R\frac{di_2}{dt} + \frac{i_2}{C} \tag{5}$$

Adding equations (4) and (5) and distributing the differentiations (which can be done only when the derivatives are of the first degree) we obtain

$$0 = L\frac{d^2(i_1 + i_2)}{dt^2} + R\frac{d(i_1 + i_2)}{dt} + \frac{(i_1 + i_2)}{C} \tag{6}$$

whence it appears that the sum of i_1 and i_2 substituted in the original equation satisfies it; that is, the sum of the solutions is a solution, as was to be proved.

If we have a third solution it can be combined with the sum of the first two solutions, just as the first solution was combined with the second so that the sum of any number of solutions is a solution.

Note 2. The Sum of Multiples of Several Solutions of a Homogeneous Linear Differential Equation is a Solution.—If $i = i_1$ is a

solution, equation (4) is true. Multiplying equation (4) through by any quantity A_1, we obtain

$$0 = A_1 L \frac{d^2 i_1}{dt^2} + A_1 R \frac{di_1}{dt} + \frac{A_1 i_1}{C} \tag{7}$$

and, if A_1 is independent of t (*i.e.*, a constant) we may introduce it within the sign of differentiation (only provided all the derivatives enter only to the first degree) and obtain

$$0 = L \frac{d^2(A_1 i_1)}{dt^2} + R \frac{d(A_1 i_1)}{dt} + \frac{A_1 i_1}{C} \tag{8}$$

which is our original equation (1) with $A_1 i_1$ substituted for i. Therefore, $i = A_1 i_1$ is a solution of (1).

Likewise, if $i = i_2$ is a solution, it can be proved that $A_2 i_2$ is a solution, and by the proposition above their sum is a solution.

The conclusion is this. *If we have a linear, homogeneous differential equation with constant coefficients, and we find several solutions of the equation, we may take any number of the solutions, multiply each by any arbitrary constant and add together the multiples and obtain thereby a result which is a solution of the original differential equation.*

Note 3. Proof that the Number of Independent Arbitrary Constants in the Solution of a Differential Equation Cannot be Greater than the Order of the Differential Equation.—As a first step toward the proof of this proposition, let us consider the formation of some differential equations by the elimination of constant from a relation between the dependent variable, the independent variable, and the arbitrary constants.

Example 1. Given

$$y = Ax \tag{9}$$

in which A is an arbitrary constant; to form an equivalent differential relation between y and x, not containing A. This can be done by the elimination of A between (9) and its derivative equation. Only one derivative equation is necessary; namely, the equation obtained by taking the first derivative of (9). This derivative equation is

$$\frac{dy}{dx} = A \tag{10}$$

Eliminating A between (9) and (10) we obtain

$$x \frac{dy}{dx} - y = 0 \tag{11}$$

The differential equation (11) is an equation of the first order. The order can be raised further by differentiation. The number of arbitrary constants does not fix the order of the differential equation, but the number of arbitrary constants determines the minimum order of the resulting differential equation. The differential equation cannot be of an order lower than the first, when the solution contains one arbitrary constant, for in order to eliminate the constant two equations are required—the given equation (9) and some derivative, which results in a differential equation of order at least as high as the first.

Example 2. Given

$$y = A_1e^{k_1t} + A_2e^{k_2t} + A_3e^{k_3t} \tag{12}$$

in which t is the independent variable, and A_1, A_2, and A_3 are arbitrary constants, to form a differential equation of which (12) is a solution. To eliminate the three arbitrary constants, four equations are necessary: for example, the equation (12) and three equations obtained by taking successive derivatives of (12). The successive derived equations are

$$\frac{dy}{dt} = A_1k_1e^{k_1t} + A_2k_2e^{k_2t} + A_3k_3e^{k_3t} \tag{13}$$

$$\frac{d^2y}{dt^2} = A_1k_1{}^2e^{k_1t} + A_2k_2{}^2e^{k_2t} + A_3k_3{}^2e^{k_3t} \tag{14}$$

$$\frac{d^3y}{dt^3} = A_1k_1{}^3e^{k_1t} + A_2k_2{}^3e^{k_2t} + A_3k_3{}^3e^{k_3t} \tag{15}$$

Now an elimination of the arbitrary constants from (12), (13), (14) and (15) gives

$$\frac{d^3y}{dt^3} - (k_1 + k_2 + k_3)\frac{d^2y}{dt^2} + (k_1k_2 + k_1k_3 + k_2k_3)\frac{dy}{dt} - k_1k_2k_3y = 0 \tag{16}$$

which is a differential equation of the third order.

It is apparent that the three constants of (12) cannot be eliminated without using at least three of the derived equations, and arriving at a differential equation of at least the third order.

In like manner, if we have a functional relation containing n arbitrary independent constants, and we eliminate the constants by using the derived equations, we shall finally arrive at a differential equation of at least the nth order.

We have said *at least* the nth order, for it is apparent that, if

we had wished, we might have used higher derivatives than the nth in order to eliminate the n constants.

The conclusion is: *The solution of a differential equation cannot contain more arbitrary, independent constants than the order of the differential equation.*

Note 4. A Solution Containing n Independent Arbitrary Constants is the Most General Solution of a Linear, Differential Equation of the nth Order with Constant Coefficients, and Embraces Every Other Solution as a Special Case, Obtainable by Giving Specific Values to the Constants.—We shall prove this proposition first for the case in which the differential equation is homogeneous. Taking t for the independent variable and y for the dependent variable let

$$y = A_1 f_1(t) + A_2 f_2(t) + \ldots A_n f_n(t) \tag{17}$$

be a solution of a linear, homogeneous differential equation of the nth order, and let this solution contain n arbitrary, independent constants $A_1, A_2, \ldots A_n$. To prove that any other function

$$y = f_r(t) \tag{18}$$

cannot be a solution unless derivable from (17) by giving proper values to some of the constants. For if there is such a solution (18), then

$$y = A_1 f_1 + A_2 f_2 + \ldots A_n f_n + A_r f_r \tag{19}$$

is a solution by Note 2, where A_r is a new arbitrary constant. But by Note 3 this cannot be for it is impossible to have in the solution more independent arbitrary constants than the order of the equation. Therefore, (18) cannot be a solution unless it be a special case of (17). It may be such a special case, for in that case it would not bring with it a new arbitrary constant A_r.

The proof thus far holds only provided the linear, differential equation is also homogeneous, for only in case of the homogeneous linear equation does the proposition of the additivity of multiples of solutions (Note 2) apply.

Next let us treat the case in which the original linear, differential equation is not homogeneous. The general form of this equation may be written

$$Py + P_1 \frac{dy}{dt} + P_2 \frac{d^2y}{dt^2} + \ldots + P_n \frac{d^ny}{dt^n} = f(t) \tag{20}$$

in which P, P_1, P_2, P_n are constant coefficients. For reference let us write down the equation

$$Py + P_1\frac{dy}{dt} + P_2\frac{d^2y}{dt^2} + \quad . \quad . \quad + P_n\frac{d^ny}{dt^n} = 0 \tag{21}$$

Suppose that we have a solution of (20) containing n independent arbitrary constants, $A_1, A_2, \ldots A_n$, in the general form

$$y_1 = A_1f_1 + A_2f_2 + \quad . \quad . \quad + A_nf_n \tag{22}$$

in which $f_1, f_2, \ldots f_n$ are functions of t. If there is any other solution of (20) not comprehended in (22), let it be

$$y_2 = f_r(t) \tag{23}$$

If (22) and (23) are both solutions of(20), then

$$y = y_1 - y_2 \tag{24}$$

is a solution of (21), for y_1 reduces the left-hand member of (20) to $f(t)$, and y_2 reduces the left-hand member of (20) to the same $f(t)$; and by subtraction $y = y_1 - y_2$ reduces this member to 0, and therefore satisfies (21).

Also by Note 2,

$$y = A_r(y_1 - y_2) \tag{25}$$

where A_r is any arbitrary constant, is a solution of (21). That is

$$y = A_rA_1f_1 + A_rA_2f_2 + \quad . \quad . \quad . \quad + A_rA_nf_n + A_rf_r \tag{26}$$

must be a solution of (21); which is impossible, because it contains $n + 1$ arbitrary constants, unless $f_r(t)$ is a special case of y_1. We have the result that if we have of equation (20) any solution containing n arbitrary independent constants it is the general solution, and contains any other solutions as a special case obtainable by giving specific values to some of the arbitrary constants.

Whether the original linear, differential equation is homogeneous or not, we have proved the proposition stated at the head of this note.

When the equations are not linear it is proved in books on differential equations that the general solution of the nth order equation has n arbitrary constants but that there are certain *singular solutions* which are not derivable from the general solution by giving specific values to the arbitrary constants.

In employing the criterion of this note as a test of the generality of the solution, care must be taken to ascertain that the n arbitrary constants are independent. If they are not independent the solution is not the general solution.

Note 5. General solution of the equation

$$\frac{di}{dt} + pi = f(t) \tag{27}$$

where p is a constant.

For reference write down the equation

$$\frac{di}{dt} + pi = 0 \tag{28}$$

Let $i = T_2$ be any solution of (28), where T_2 is a function of t. If we indicate the time derivatives of T_2 by T'_2, we shall have by (28)

$$T'_2 + pT_2 = 0 \tag{29}$$

Now let the complete solution of (27) be written in the form

$$i = T_1T_2 \tag{30}$$

where T_1 is also a function of t. Then by (27)

$$T'_1T_2 + T_1T'_2 + pT_1T_2 = f(t) \tag{31}$$

whence by (29)

$$T'_1T_2 = f(t) \tag{32}$$

Integrating we obtain

$$T_1 = \int \frac{f(t)\,dt}{T_2} + A.$$

Therefore, by (30)

$$i = T_2\left[\int \frac{f(t)\,dt}{T_2} + A\right] \tag{33}$$

Now T_2 is any solution of (28). The simplest solution may be used, and (33) will still be true. The simplest T_2 that is a solution of (28) is

$$T_2 = e^{-pt}.$$

This substituted in (33) gives

$$i = Ae^{-pt} + e^{-pt}\int e^{pt} f(t)\,dt \tag{34}$$

In performing the integration indicated in equation (34) no constant of integration is to be added, since the only arbitrary allowable for the solution of a first order equation is already comprised in A.

Equation (34) *is the complete integral, or general solution, of* (27).

Note 6. General solution of the equation

$$L\frac{di}{dt} + Ri + \int\frac{idt}{C} = v = f(t) \tag{35}$$

where L, R, and C are constants.

Differentiating, we obtain

$$\dot{v} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} \tag{36}$$

in which $\dot{v}$ is the time derivative of v.

Replacing i in (35) by $\frac{dq}{dt}$ we obtain also

$$v = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \tag{36a}$$

For reference write down the auxiliary equation

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} \tag{37}$$

We have seen in early chapters of the text that $i = e^{kt}$ is a particular solution of (37), where

$$k = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a + j\omega \tag{38}$$

where

$$a = R/2L$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Let us now write the solution of (36) in the form of

$$i = Te^{kt}, \tag{39}$$

where T is some function of t, and substitute this solution directly in (36). We obtain, after division by e^{kt},

$$\frac{\dot{v}}{e^{kt}} = L\ddot{T} + (2kL + R)\dot{T} \tag{40}$$

where $\dot{T}$ and $\ddot{T}$ are the first and second time derivatives of T. The simplicity of this equation arises from the fact that

$$0 = T\left\{L\frac{d^2}{dt^2}(e^{kt}) + R\frac{d}{dt}(e^{kt}) + \frac{1}{C}(e^{kt})\right\} \tag{41}$$

because $i = e^{kt}$ is a solution of (37) when k has the value given in (38).

Equation (40) which we have derived from (35), when integrated, gives, after division by L,

$$\dot{T} + \left(2k + \frac{R}{L}\right) T = B_1 + \frac{1}{L}\int ve^{-kt}dt \tag{42}$$

where B_1 is a constant of integration. By (38) the coefficient of T is $2j\omega$. This equation is of the form of (27) and by (34) gives

$$T = A_1e^{-2j\omega t} + e^{-2j\omega t}\int\left\{B_1e^{2j\omega t} + \frac{e^{2j\omega t}}{L}\int ve^{-kt}\,dt\right\}dt \tag{43}$$

Integrating the B_1 term and replacing $B_1/2j\omega$ by A_2 we obtain

$$T = A_1e^{-2j\omega t} + A_2 + \frac{e^{-2j\omega t}}{L}\int\left[e^{2j\omega t}\int ve^{+at-j\omega t}\,dt\right]dt$$

and since

$$i = e^{kt}T_1 = e^{-at+j\omega t}T_1,$$

we have

$$i = e^{-at}\left\{A_1e^{-j\omega t} + A_2e^{+j\omega t}\right\} + \frac{e^{-at-j\omega t}}{L}\int\left[e^{2j\omega t}\int \dot{v}e^{at-j\omega t}\,dt\right]dt \tag{44}$$

The integration indicated in the last term can be carried one step farther by integration by parts

$$\int udv = uv - \int vdu$$

$$dv = e^{2j\omega t}\,dt, \qquad v = \frac{e^{2j\omega t}}{2j\omega}$$

$$u = \int ve^{at-j\omega t}\,dt, \quad du = \dot{v}e^{at-j\omega t}\,dt$$

whence

$$\int udv = \frac{e^{2j\omega t}}{2j\omega}\int \dot{v}e^{at-j\omega t}\,dt - \frac{1}{2j\omega}\int \dot{v}e^{at+j\omega t}\,dt$$

Therefore (44) becomes

$$i = e^{-at}\left\{A_1e^{-j\omega t} + A_2e^{+j\omega t}\right\}$$
$$+ \frac{e^{-at}}{2jL\omega}\left\{e^{+j\omega t}\int \dot{v}e^{at-j\omega t}\,dt - e^{-j\omega t}\int \dot{v}e^{at+j\omega t}\,dt\right\} \tag{45}$$

where

$$a = \frac{R}{2L},$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

In equation (45) A and B are arbitrary constants, and no further arbitrary constants are to be introduced after the indicated integrations are performed.

Equation (45) *is the formal solution of the differential equation* (35), *and gives* i *directly when* v *is given as a function of* t, *provided the indicated integrations can be performed.*

It is evident from a comparison of (36a) with (36) that the solution for q differs from that for i (45) only in having different arbitrary constants and in having v replaced by v, giving

$$q = e^{-at}\left\{B_1 e^{-j\omega t} + B_2 e^{j\omega t}\right\} + \frac{e^{-at}}{2jL\omega}\left\{e^{j\omega t}\int v e^{at-j\omega t}\,dt - e^{-j\omega t}\int v e^{at+j\omega t}\,dt\right\} \qquad (46)$$

Equation (46) *is the formal solution of the differential equation* (36a), *and gives* q *when* v *is known as a function of* t, *provided the indicated integrations can be performed.* B_1 *and* B_2 *are arbitrary constants and no further arbitrary constants are to be introduced after performing the indicated integrations.*

Expression of i in terms of v instead of $\dot{v}$.—The integrations indicated in equation (45) may be performed by parts in such a way as to replace v by v. This is done as follows:

$$\int v e^{at-j\omega t}\,dt = uv - \int v du,$$

where

$$dv = v dt, \quad u = e^{at-j\omega t}$$

$$= v e^{at-j\omega t} - (a - j\omega)\int v e^{+at-j\omega t}\,dt$$

Likewise

$$\int \dot{v} e^{at+j\omega t}\,dt = v e^{at+j\omega t} - (a + j\omega)\int v e^{at+j\omega t}\,dt$$

whence (45) becomes

$$i = e^{-at}\left\{A_1 e^{-j\omega t} + A_2 e^{j\omega t}\right\} + \frac{e^{-at}}{2jL\omega}\left\{(a + j\omega)e^{-j\omega t}\int v e^{at+j\omega t}\,dt - (a - j\omega)e^{j\omega t}\int v e^{at-j\omega t}\,dt\right\} (47)$$

Further Transformation of Equations (46) and (47).—We may now change the expressions for q and i into definite integrals with the constants explicitly determined by the following process, taking (47) as a sample. We may write the identity, employing a change of variable,

$$\int v e^{at+j\omega t}\,dt \equiv \left[\int v_{t'} e^{at'+j\omega t'}\,dt'\right]_{t'=t}$$

where $v_{t'}$ means v (which is a function of t) with its t everywhere replaced by t'.

If now on the right-hand side we add and subtract the same quantity, we obtain

$$\int ve^{at+j\omega t}dt \equiv \int_{t'=0}^{t'=t} v_{t'}e^{at'+j\omega t'}dt' + \left[\int v_{t'}e^{at'+j\omega t'}dt'\right]_{t'=0}$$

This last term is a constant, which when introduced into (47) will merely change the constant A_1 to A'_1 say.

Making a similar transformation of the last integral of (47), it is to be noted that in (47) the multipliers of the resulting integrals may be introduced under the integral signs, since the integrations are now with respect to t' instead of t. So that (47) becomes

$$i = e^{-at}\left\{A'_1e^{-j\omega t} + A'_2e^{j\omega t}\right\}$$

$$+ \frac{a + j\omega}{2jL\omega}\int_{t'=0}^{t'=t} v_{t'}e^{-a(t-t')}e^{+j\omega(t'-t)}\,dt' - \frac{a - j\omega}{2jL\omega}\int_{t'=0}^{t'=t} v_{t'}e^{-a(t-t')}e^{-j\omega(t'-t)}\,dt' \qquad (48)$$

This now becomes by changing to trigonometric function

$$i = Ie^{-at}\sin(\omega t + \phi_1)$$

$$+ \frac{1}{L}\left[\int_{t'=0}^{t'=t} v_{t'}e^{-a(t-t')}\left\{\cos\omega(t-t') - \frac{a}{\omega}\sin\omega(t - t')\right\}\right. \qquad (48)$$

or

$$i = Ie^{-at}\sin(\omega t + \phi_1) + \frac{\sqrt{\omega^2 + a^2}}{L\omega}\int_{t'=0}^{t'=t} v_{t'}e^{-a(t-t')}\cos\left\{\omega(t-t') + \tan^{-1}\frac{a}{\omega}\right\}dt' \qquad (50)$$

By a similar treatment of (46), we obtain

$$q = Qe^{-at}\sin(\omega t + \phi_2) + \frac{1}{L\omega}\int_{t'=0}^{t_1=t} v_{t'}e^{-a(t-t')}\sin\omega(t-t')\,dt' \qquad (51)$$

There are relations between the Q and ϕ_2 of (51) and the I and ϕ_1 of (51). These relations may be obtained by equating $\frac{dq}{dt}$ to i. We obtain, by differentiating the q of (51) [see Byerly Integral Calculus equation (6), p. 95].

$$i = Qe^{-at}\Big\{- a \sin(\omega t + \varphi_2) + \omega \cos(\omega t + \varphi_2)$$

$$+ \frac{1}{L\omega}\int_{t_1=0}^{t_1=t} v_{t'}e^{-a(t-t')}\Big\{- a \sin \omega\,(t-t')dt' + \omega \cos \omega\,(t-t')\,dt'\Big\}$$

This compared with (49) shows that

$$I \sin(\omega t + \varphi_1) = Q\,\{\omega \cos(\omega t + \varphi_2) - a \sin(\omega t + \varphi_2)\},$$

and this equation is true for all values of t.

Letting $\omega t = -\varphi_1$ we have

$$\tan(\phi_2 - \varphi_1) = \frac{\omega}{a} \tag{52}$$

and letting $\omega t = -\varphi_2$ we obtain

$$= - Q\sqrt{\omega^2 + a^2} \tag{53}$$

These relations put into (51) give

$$q = \frac{-I}{\sqrt{\omega^2 + a^2}}\,e^{-at}\sin\left(\omega t + \phi_1 + \tan^{-1}\frac{\omega}{a}\right)$$

$$+ \frac{1}{L\omega}\int_{t'=0}^{t'=t} v_{t'}e^{-a(t-t')}\sin \omega\,(t-t')\,dt' \tag{54}$$

If now we introduce into (50) and (54) the initial conditions (say)

$$t = 0, \quad i = I_0, \quad q = Q_0 \tag{55}$$

we have, since the upper and lower limits of the definite integrals become identical,

$$\left.\begin{aligned} I_0 &= I \sin \phi \\ Q_0 &= \frac{-I}{\omega^2 + a^2}(a \sin \phi_1 + \omega \cos \phi_1) \end{aligned}\right\} \tag{56}$$

Dividing Q_0 by I_0 we obtain

$$\left.\begin{aligned} \frac{Q_0}{I_0} &= -\frac{a}{\omega^2 + a^2} + \frac{\omega \cot \phi_1}{\omega^2 + a^2} \\ \phi_1 &= \cot^{-1}\left\{\frac{Q_0}{I_0}\,\frac{\omega^2 + a^2}{\omega} + \frac{a}{\omega}\right\} \end{aligned}\right\} \tag{57}$$

whence

$$\sin \phi_1 = \frac{1}{\sqrt{1 + \left\{\frac{a^2 + \omega^2}{\omega}\,\frac{Q_0}{I_0} + \frac{a}{\omega}\right\}^2}}.$$

Therefore,

$$I = I_0\sqrt{1 + \left\{\frac{a^2 + \omega^2}{\omega}\frac{Q_0}{I_0} + \frac{a}{\omega}\right\}^2} \tag{58}$$

Equations (50) *and* (54) *give the required values of i and q where the constants, I and ϕ_1 have the values given in* (57) *and* (58). *In these equations I_0 and Q_0 are the values of current and charge, respectively, at the time $t = 0$. Note.—In case $I_0 = Q_0 = 0$* (57) *and* (58)) *become indeterminate, but* (56) *shows that in that case $I = 0$.*

Note 7. Solution of the equation

$$0 = \mathbf{L}\frac{d^2\mathbf{i}}{dt^2} + \mathbf{R}\frac{d\mathbf{i}}{dt} + \frac{\mathbf{i}}{\mathbf{C}} \tag{59}$$

in the Critical Case in which

$$\mathbf{R}^2 = 4\mathbf{L}/\mathbf{C} \tag{60}$$

In view of equation (60), equation (59) may be written

$$0 = \frac{d^2i}{dt^2} + \frac{Rdi}{Ldt} + \frac{R^2i}{4L^2} \tag{61}$$

This equation may be reduced to one of a lower order by separating $\frac{Rdi}{Ldt}$ into $\frac{Rdi}{2Ldt} + \frac{Rdi}{2Ldt}$, and indicating operations as follows:

$$0 = \frac{d}{dt}\left(\frac{di}{dt} + \frac{Ri}{2L}\right) + \frac{R}{2L}\left(\frac{di}{dt} + \frac{Ri}{2L}\right).$$

Whence

$$\frac{d\left(\frac{di}{dt} + \frac{Ri}{2L}\right)}{\frac{di}{dt} + \frac{Ri}{2L}} = -\frac{Rdt}{2L}.$$

Integrating, we obtain

$$\log\left(\frac{di}{dt} + \frac{Ri}{2L}\right) = -\frac{Rt}{2L} + B \tag{62}$$

in which B is a constant of integration Let $B = \log A_2$; A_2 being an arbitrary constant, then (62) gives

$$\frac{di}{dt} + \frac{Ri}{2L} = A_2e^{-\frac{Rt}{2L}},$$

which is of the first order, and may be integrated by the use of the formal equation (34) of Note 5, giving

$$i = A_1 e^{-\frac{Rt}{2L}} + e^{-\frac{Rt}{2L}} \int e^{+\frac{Rt}{2L}} A_2 e^{-\frac{Rt}{2L}} dt$$

and, therefore,

$$i = (A_1 + A_2 t) e^{-\frac{Rt}{2L}} \qquad (63)$$

in which A_1 and A_2 are arbitrary constants of integration.

Equation (63) *is the complete integral, or general solution of* (59) *in the Critical Case.*

Note 8. Solution of the equation

$$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \qquad (64)$$

in which V is a constant.

The solution of this equation may be obtained directly by substituting q for i and V for v in equation (45) of Note 6.

A more elementary method of solving (56) is by inserting a new variable $z = q - CV$, when (64) becomes

$$0 = L\frac{d^2z}{dt^2} + R\frac{dz}{dt} + \frac{z}{C} \qquad (65)$$

which has already been solved (see Chapter II) with the following results:

In general	In critical case
$z = B_1 e^{k_1 t} + B_2 e^{k_2 t}$ where $k_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{Lc}}$ and $k_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{Lc}}$ whence by the value of z $q = B_1 e^{k_1 t} + B_2 e^{k_2 t} + CV$	$z = (B_1 + B_2 t) e^{-\frac{Rt}{2L}}$ This is the solution in case $R^2 = 4L/C$ whence by the value of z $q = (B_1 + B_2 t) e^{-\frac{Rt}{2L}} + CV$

Table I[1]

Relation of Capacity-inductance Product to Undamped Wavelength and Frequency of a Circuit, Together with Squares of Wavelengths

Units.—

λ is in meters,
n is in cycles per second,
L is in micro-henries,
C is in microfarads.

Formulas Employed in Calculation.—

$$\lambda = 3 \times 10^8 \times 2\pi\sqrt{LC} \times 10^{-6} \tag{1}$$

This last factor comes from the fact that a micro-henry is 10^{-6} henries, and a microfarad is 10^{-6} farads. The product involves 10^{-12}, of which the square root is 10^{-6}.

By squaring and transposing equation (1), we obtain

$$L \times C = 28145\lambda^2 \times 10^{-11} \tag{2}$$

In computing n, the formula employed is

$$n = (3 \times 10^8) \div \lambda \tag{3}$$

Accuracy.—The values in the table were computed and checked on a calculating machine and are accurate to the last figure given.

Rule for Extending Range of the Table.—If we annex one zero at the end of wavelength values,

(*a*) we must annex two zeros to values of λ^2,
(*b*) omit the last digit from values of n,
(*c*) displace decimal point two places to right in the $L \times C$ values.

[1] A table of this character prepared by Mr. Greenleaf W. Pickard has been issued by the Wireless Specialty Apparatus Company of Boston. Mr. Pickard's table has only three significant figures in values of $L \times C$, and four significant figures in values of n. The utility of Mr. Pickard's table has led me to compute and publish the present table, which is augmented by the inclusion of the λ^2 values, and which is accurate presumably to all of the figures given.

λ	λ^2	$L \times C$	n	λ	λ^2	$L \times C$	n
100	10000	0 0028145	3000000	136	18496	0 0052075	2205882
101	10201	0 0028711	2970297	137	18769	0 0052825	2189781
102	10404	0 0029282	2941177	138	19044	0 0053599	2173913
103	10609	0 0029859	2912621	139	19321	0 0054379	2158274
104	10816	0 0030442	2884616	140	19600	0 0055164	2142857
105	11025	0 0031030	2857143	141	19881	0 0055955	2127660
106	11236	0 0031624	2830189	142	20164	0 0056752	2112676
107	11449	0 0032223	2803738	143	20449	0 0057554	2097902
108	11664	0 0032828	2777778	144	20736	0 0058361	2083333
109	11881	0 0033439	2752294	145	21025	0 0059175	2068966
110	12100	0 0034055	2727272	146	21316	0 0059994	2054795
111	12321	0 0034677	2702703	147	21609	0 0060819	2040816
112	12544	0 0035305	2678571	148	21904	0 0061649	2027027
113	12769	0 0035938	2654867	149	22201	0 0062485	2013423
114	12996	0 0036577	2631579	150	22500	0 0063326	2000000
115	13225	0 0037222	2608696	151	22801	0 0064173	1986755
116	13456	0 0037872	2586207	152	23104	0.0065026	1973684
117	13689	0 0038528	2564103	153	23409	0 0065885	1960784
118	13924	0 0039189	2542373	154	23716	0 0066749	1948052
119	14161	0 0039856	2521008	155	24025	0 0067618	1935484
120	14400	0.0040529	2500000	156	24336	0 0068494	1923077
121	14641	0.0041207	2479339	157	24649	0 0069375	1910828
122	14884	0 0041891	2459016	158	24964	0 0070271	1898734
123	15129	0 0042581	2439024	159	25281	0 0071153	1886792
124	15376	0 0043276	2419355	160	25600	0 0072051	1875000
125	15625	0 0043977	2400000	161	25921	0 0072955	1863354
126	15876	0.0044683	2380952	162	26244	0.0073864	1851852
127	16129	0.0045395	2362205	163	26569	0 0074778	1840491
128	16384	0 0046113	2343750	164	26896	0 0075699	1829268
129	16641	0.0046836	2325581	165	27225	0.0076625	1818182
130	16900	0.0047565	2307692	166	27556	0.0077556	1807229
131	17161	0.0048300	2290076	167	27889	0 0078494	1796407
132	17424	0.0049040	2272727	168	28224	0.0079436	1785714
133	17689	0 0049786	2255639	169	28561	0.0080385	1775148
134	17956	0.0050537	2238806	170	28900	0.0081339	1764706
135	18225	0.0051294	2222222	171	29241	0.0082299	1754386

λ	λ^2	$L \times C$	n	λ	λ^2	$L \times C$	n
172	29584	0 0083264	1744186	216	46656	0 0131313	1388889
173	29929	0 0084235	1734104	218	47524	0 0133756	1376147
174	30276	0 0085212	1724138	220	48400	0 0136222	1363636
175	30625	0 0086194	1714286	222	49284	0 0138710	1351352
176	30976	0 0087182	1704545	224	50176	0 0141220	1339286
177	31329	0 0088175	1694915	226	51076	0 0143753	1327434
178	31684	0 0089175	1685393	228	51984	0 0146309	1315790
179	32041	0 0090179	1675978	230	52900	0 0148887	1304348
180	32400	0 0091190	1666667	232	53824	0 0151488	1293104
181	32761	0 0092206	1657459	234	54756	0 0154111	1282051
182	33124	0 0093227	1648352	236	55696	0 0156756	1271186
183	33489	0 0094255	1639344	238	56644	0 0159425	1260504
184	33856	0 0095288	1630435	240	57600	0.016212	1250000
185	34225	0 0096326	1621622	242	58564	0 016483	1239669
186	34596	0 0097370	1612903	244	59536	0 016756	1229508
187	34969	0 0098420	1604278	246	60516	0 017032	1219512
188	35344	0.0099476	1595745	248	61504	0 017310	1209677
189	35721	0.0100537	1587302	250	62500	0 017591	1200000
190	36100	0 0101603	1578947	252	63504	0 017873	1190476
191	36481	0 0102676	1570681	254	64516	0 018158	1181102
192	36864	0 0103754	1562500	256	65536	0.018445	1171875
193	37249	0.0104837	1554404	258	66564	0.018734	1162791
194	37636	0.0105927	1546392	260	67600	0 019026	1153846
195	38025	0.0107021	1538462	262	68644	0.019320	1145038
196	38416	0.0108122	1530612	264	69696	0.019616	1136364
197	38809	0 0109228	1522843	266	70756	0.019914	1127819
198	39204	0.0110340	1515152	268	71824	0 020215	1119403
199	39601	0 0111457	1507538	270	72900	0.020518	1111111
200	40000	0 0112580	1500000	272	73984	0.020823	1102941
202	40804	0.0114843	1485149	274	75076	0.021130	1094891
204	41616	0.0117128	1470588	276	76176	0.021440	1086956
206	42436	0 0119436	1456311	278	77284	0.021752	1079137
208	43264	0 0121767	1442308	280	78400	0.022066	1071429
210	44100	0.0124119	1428572	282	79524	0.022382	1063830
212	44944	0 0126495	1415094	284	80656	0.022701	1056338
214	45796	0 0128893	1401869	286	81796	0.023021	1048951

λ	λ^2	$L \times C$	n	λ	λ^2	$L \times C$	n
288	82944	0 023345	1041667	360	129600	0 036476	833333
290	84100	0 023670	1034483	362	131044	0 036881	828729
292	85264	0 023998	1027397	364	132496	0.037292	824176
294	86436	0 024327	1020408	366	133956	0 037703	819672
296	87616	0 024660	1013513	368	135424	0 038114	815217
298	88804	0 024994	1006712	370	136900	0 038531	810811
300	90000	0 025331	1000000	372	138384	0 038947	806452
302	91204	0 025669	993377	374	139876	0 039369	802139
304	92416	0 026010	986842	376	141376	0 039791	797872
306	93636	0 026354	980392	378	142884	0 040214	793651
308	94864	0 026699	974026	380	144400	0 040641	789474
310	96100	0 027047	967742	382	145924	0 041069	785340
312	97344	0 027397	961538	384	147456	0 041503	781250
314	98596	0 027750	955414	386	148996	0 041936	777202
316	99856	0.028104	949367	388	150544	0 042369	773196
318	101124	0 028460	943396	390	152100	0 042809	769231
320	102400	0 028820	937500	392	153664	0.043248	765306
322	103684	0.029181	931677	394	155236	0 043692	761421
324	104976	0 029547	925926	396	156816	0 044137	757576
326	106276	0.029913	920246	398	158404	0 044582	753769
328	107584	0.030278	914634	400	160000	0 045032	750000
330	108900	0 030650	909091	402	161604	0.045482	746269
332	110224	0 031021	903614	404	163216	0.045938	742574
334	111566	0 031401	898204	406	164836	0 046394	738916
336	112896	0 031776	892857	408	166464	0.046850	735294
338	114244	0.032153	887574	410	168100	0 047312	731706
340	115600	0.032536	882353	412	169744	0.047773	728155
342	116964	0.032918	877193	414	171396	0.048241	724638
344	118336	0.033307	872093	416	173056	0.048708	721154
346	119716	0.033695	867052	418	174724	0.049175	717703
348	121104	0.034084	862069	420	176400	0 049648	714286
350	122500	0.034478	857143	422	178084	0 050121	710900
352	123904	0.034872	852273	424	179776	0 050599	707547
354	125316	0.035271	847458	426	181476	0.051078	704225
356	126736	0.035671	842697	428	183184	0.051556	700935
358	128164	0 036071	837989	430	184900	0 052040	697674

λ	$λ^2$	$L \times C$	n	λ	$λ^2$	$L \times C$	n
432	186624	0 052524	694445	510	260100	0 073205	588235
434	188356	0 053014	691244	515	265225	0 074649	582524
436	190096	0 053504	688073	520	270400	0 076104	576923
438	191844	0 053993	684932	525	275625	0 077576	571429
440	193600	0 054489	681818	530	280900	0 079059	566038
442	195364	0.054984	678733	535	286225	0 080559	560748
444	197136	0 055485	675676	540	291600	0 082071	555556
446	198916	0 055986	672646	545	297025	0 083599	550459
448	200704	0 056487	669643	550	302500	0 085139	545455
450	202500	0 056994	666667	555	308025	0 086695	540541
452	204304	0 057500	663717	560	313600	0 088263	535714
454	206116	0 058012	660793	565	319225	0 089847	530974
456	207936	0 058525	657895	570	324900	0 091443	526316
458	209764	0 059037	655022	575	330625	0.093056	521739
460	211600	0 059555	652174	580	336400	0 094680	517241
462	213444	0 060073	649351	585	342225	0 096321	512821
464	215296	0 060596	646552	590	348100	0 097973	508475
466	217156	0 061120	643777	595	354025	0 099642	504202
468	219024	0 061643	641026	600	360000	0 10132	500000
470	220900	0 062172	638298	605	366025	0 10302	495868
472	222784	0 062701	635593	610	372100	0.10473	491803
474	224676	0 063236	632912	615	378225	0.10645	487805
476	226576	0 063771	630252	620	384400	0 10819	483871
478	228484	0 064306	627615	625	390625	0.10994	480000
480	230400	0 064846	625000	630	396900	0.11171	476191
482	232324	0 065386	622407	635	403225	0.11349	472441
484	234256	0 065932	619835	640	409600	0.11528	468750
486	236196	0 066478	617284	645	416025	0 11709	465116
488	238144	0 067025	614754	650	422500	0.11891	461539
490	240100	0 067576	612245	655	429025	0.12075	458015
492	242064	0 068128	609756	660	435600	0.12260	454545
494	244036	0.068685	607287	665	442225	0.12447	451128
496	246016	0.069242	604839	670	448900	0.12634	447761
498	248004	0 069800	602410	675	455625	0.12824	444444
500	250000	0.070363	600000	680	462400	0.13014	441176
505	255025	0 071778	594059	685	469225	0.13206	437956

λ	λ^2	$L \times C$	n	λ	λ^2	$L \times C$	n
690	476100	0 13400	434783	870	756900	0 21303	344828
695	483025	0 13595	431655	875	765625	0 21549	342857
700	490000	0 13791	428571	880	774400	0 21795	340909
705	497025	0 13989	425532	885	783225	0 22044	338983
710	504100	0 14188	422535	890	792100	0 22294	337079
715	511225	0 14389	419580	895	801025	0 22545	335195
720	518400	0 14590	416667	900	810000	0 22797	333333
725	525625	0 14794	413793	905	819025	0 23052	331492
730	532900	0 14998	410959	910	828100	0 23307	329670
735	540225	0 15205	408163	915	837225	0 23564	327869
740	547600	0 15412	405405	920	846400	0 23822	326087
745	555025	0 15621	402685	925	855625	0 24082	324324
750	562500	0.15832	400000	930	864900	0 24343	322581
755	570025	0.16043	397351	935	874225	0 24605	320856
760	577600	0 16257	394737	940	883600	0 24869	319149
765	585225	0.16471	392157	945	893025	0 25134	317460
770	592900	0 16687	389610	950	902500	0 25401	315790
775	600625	0.16905	387097	955	912025	0 25669	314136
780	608400	0 17123	384615	960	921600	0 25938	312500
785	616225	0 17344	382166	965	931225	0.26209	310881
790	624100	0.17565	379747	970	940900	0.26482	309278
795	632025	0 17788	377359	975	950625	0 26755	307693
800	640000	0 18013	375000	980	960400	0 27030	306122
805	648025	0.18239	372671	985	970225	0 27307	304568
810	656100	0 18466	370370	990	980100	0 27585	303030
815	664225	0.18695	368098	995	990025	0 27864	301508
820	672400	0 18925	365854	1000	1000000	0 28145	300000
825	680625	0.19156	363636	1005	1010025	0 28427	298507
830	688900	0 19389	361446	1010	1020100	0 28711	297030
835	697225	0.19624	359282	1015	1030225	0 28996	295567
840	705600	0.19859	357143	1020	1040400	0.29282	294118
845	714025	0.20096	355030	1025	1050625	0 29569	292683
850	722500	0.20335	352941	1030	1060900	0 29859	291262
855	731025	0.20575	350877	1035	1071225	0 30149	289855
860	739600	0.20816	348837	1040	1081600	0 30442	288462
865	748225	0.21059	346821	1045	1092025	0.30734	287081

λ	$λ^2$	$L \times C$	n	λ	$λ^2$	$L \times C$	n
1050	1102500	0 31030	285714	1150	1322500	0 37222	260870
1055	1113025	0 31325	284360	1155	1334025	0 37545	259740
1060	1123600	0 31624	283019	1160	1345600	0 37872	258621
1065	1134225	0 31922	281690	1165	1357225	0 38198	257511
1070	1144900	0 32223	280374	1170	1368900	0 38528	256410
1075	1155625	0 32524	279069	1175	1380625	0 38857	255319
1080	1166400	0 32828	277778	1180	1392400	0 39189	254237
1085	1177225	0 33132	276498	1185	1404225	0 39521	253165
1090	1188100	0 33439	275229	1190	1416100	0 39856	252101
1095	1199025	0 33746	273973	1195	1428025	0 40191	251046
1100	1210000	0 34055	272727	1200	1440000	0 40529	250000
1105	1221025	0 34365	271493	1205	1452025	0 40867	248963
1110	1232100	0 34677	270270	1210	1464100	0 41207	247934
1115	1243225	0 34990	269058	1215	1476225	0 41548	246914
1120	1254400	0 35305	267857	1220	1488400	0.41891	245902
1125	1265625	0 35620	266667	1225	1500625	0.42234	244898
1130	1276900	0 35938	265487	1230	1512900	0 42581	243902
1135	1288225	0 36256	264317	1235	1525225	0 42927	242915
1140	1299600	0 36577	263158	1240	1537600	0 43276	241935
1145	1311025	0 36898	262009	1245	1550025	0 43625	240964

Table II

Radiation Resistance in Ohms of Flat-top Antenna

λ_0 = natural wavelength of antenna unloaded,
λ = wavelength when loaded with inductance at base,
$$\gamma = \frac{\text{length of flat horizontal part of antenna}}{\text{total length of antenna}}.$$

λ/λ_0	Radiation resistance in ohms for γ equal								
	0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	36.60	33.30	29.70	25.50	20.30	14.70	9.70	4.90	1.200
1.1	28.00	26.00	23.90	20.00	16.00	11.90	7.60	3.80	1.070
1.2	21.80	20.20	18.80	15.80	12.40	9.00	6.00	2.90	0.940
1.3	18.20	16.90	15.10	12.60	11.20	7.20	4.90	2.40	0.810
1.4	15.10	14.00	12.20	10.50	8.60	6.10	4.00	2.00	0.700
1.5	12.80	11.70	10.40	9.00	7.30	5.20	3.30	1.70	0.600
1.6	11.00	10.00	9.00	7.80	6.30	4.40	2.80	1.40	0.500
1.7	9.50	8.60	7.60	6.70	5.40	3.70	2.50	1.20	0.400
1.8	8.30	7.70	6.70	6.00	4.70	3.20	2.20	1.10	0.330
1.9	7.40	6.80	6.20	5.30	4.20	2.90	1.90	0.90	0.240
2.0	6.50	6.10	5.50	4.80	3.80	2.70	1.70	0.75	0.180
2.2	5.20	5.00	4.60	3.90	3.00	2.20	1.40	0.57	0.160
2.4	4.40	4.20	3.80	3.20	2.50	1.80	1.20	0.48	0.140
2.6	3.80	3.50	3.10	2.70	2.10	1.50	1.00	0.42	0.120
2.8	3.30	3.00	2.60	2.30	1.80	1.30	0.86	0.37	0.100
3.0	2.80	2.50	2.20	1.90	1.50	1.10	0.74	0.33	0.090
3.2	2.50	2.30	2.00	1.70	1.30	0.92	0.64	0.29	0.080
3.4	2.20	2.00	1.80	1.60	1.10	0.84	0.55	0.25	0.072
3.6	2.00	1.90	1.60	1.40	1.00	0.77	0.47	0.22	0.066
3.8	1.75	1.70	1.40	1.30	0.94	0.71	0.39	0.19	0.060
4.0	1.62	1.50	1.30	1.10	0.88	0.66	0.31	0.16	0.055
4.5	1.30	1.21	1.05	0.89	0.75	0.54	0.26	0.12	0.042
5.0	1.00	0.92	0.80	0.68	0.63	0.42	0.22	0.09	0.032
5.5	0.78	0.73	0.65	0.56	0.53	0.36	0.19	0.08	0.025
6.0	0.61	0.54	0.49	0.44	0.43	0.29	0.16	0.07	0.019
6.5	0.48	0.45	0.41	0.38	0.35	0.25	0.14	0.07	0.015
7.0	0.38	0.36	0.33	0.32	0.28	0.22	0.12	0.06	0.013
7.5	0.32	0.31	0.29	0.28	0.25	0.19	0.11	0.06	0.013
8.0	0.28	0.27	0.25	0.23	0.22	0.17	0.10	0.05	0.012
8.5	0.26	0.25	0.23	0.21	0.19	0.15	0.09	0.05	0.012
9.0	0.25	0.22	0.20	0.18	0.16	0.13	0.08	0.05	0.012
9.5	0.24	0.20	0.19	0.17	0.15	0.12	0.08	0.05	0.011
10.0	0.22	0.18	0.17	0.15	0.13	0.11	0.07	0.04	0.011
10.5	0.21	0.16	0.15	0.14	0.12	0.10	0.07	0.04	0.010
11.0	0.20	0.14	0.13	0.12	0.11	0.09	0.06	0.04	0.010
11.5	0.19	0.13	0.12	0.11	0.10	0.08	0.06	0.04	0.009
12.0	0.18	0.12	0.11	0.10	0.09	0.07	0.05	0.03	0.009
12.5	0.16	0.11	0.10	0.09	0.08	0.07	0.05	0.03	0.008
13.0	0.15	0.10	0.09	0.09	0.08	0.06	0.05	0.03	0.008
13.5	0.14	0.09	0.08	0.08	0.07	0.06	0.04	0.03	0.007
14.0	0.12	0.08	0.07	0.07	0.06	0.05	0.04	0.02	0.007
14.5	0.11	0.08	0.07	0.06	0.06	0.05	0.04	0.02	0.006
15.0	0.10	0.07	0.06	0.06	0.05	0.04	0.03	0.02	0.006
15.5	0.08	0.06	0.06	0.05	0.05	0.04	0.03	0.02	0.005
16.0	0.06	0.06	0.06	0.05	0.04	0.04	0.03	0.02	0.005

Table III.—For the Conversion of Units—Containing the Practical Units Together With Their Values in Terms of the Two Sets of c.g.s. Units, Where $c = 3 \times 10^{10}$ cm./sec.

Unit of	Practical units	C g s units	
		Electromagnetic	Electrostatic
Quantity.	1 Coulomb =	10^{-1} =	$10^{-1} \times c = 3 \times 10^{9}$
Current .	1 Ampere =	10^{-1} =	$10^{-1} \times c = 3 \times 10^{9}$
Potential	1 Volt =	10^{8} =	$10^{8} \div c = \frac{1}{3} \times 10^{-2}$
Resistance	1 Ohm =	10^{9} =	$10^{9} \div c^2 = \frac{1}{9} \times 10^{-11}$
Capacity	1 Farad =	10^{-9} =	$10^{-9} \times c^2 = 9 \times 10^{11}$
Inductance	1 Henry =	10^{9} =	$10^{9} \div c^2 = \frac{1}{9} \times 10^{-11}$
Energy	1 Joule =	10^{7} =	10^{7} ergs
Power. . .	1 Watt =	10^{7} =	10^{7} ergs

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